Structural Properties of $m$-Step Graphs

Bachelor Thesis

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Declaration

I hereby declare that I produced this thesis without external assistance, and that no other than
the listed references have been used as sources of information.

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1 Introduction and Basic Definitions

The definition of $m$-step graphs first requires precise definitions of graphs and paths. Throughout this thesis I will only consider simple graphs; simple in this context means finite, undirected and having neither loops nor multiple edges. Thus a graph $G = (V, E)$ is a pair of disjoint sets $V = V(G)$, the vertices, and $E = E(G)$, the edges; thereby any edge $e \in E$ is a set of two distinct elements $x, y \in V$. An edge $\{x, y\} \in E$ will be written as $xy \in E$. The set of all possible simple graphs over $V$ is denoted by $G(V) = \{(V, E) \mid \forall e \in E : e \subseteq V \land |e| = 2\}$.

Paths are graphs isomorphic to $P_n = (V, E)$ with $n \in \mathbb{N}$ vertices $V = \{v_1, \ldots, v_n\}$ and edges $E = \{v_1v_2, \ldots, v_{n-1}v_n\}$. The length of a path is the number of its edges $|E| = n - 1$. Its end vertices are $v_1$ and $v_n$ and the path is called a $v_1v_n$-path. The inner vertices are $v_2, \ldots, v_{n-1}$. A path from $v_1$ to $v_n$ is often denoted by the sequence of its vertices $v_1v_2\ldots v_n$. The vertices of $P$ (and therefore its edges) are pairwise distinct, otherwise it is called a walk.

Isomorphism is denoted by $G_1 \cong G_2$, subgraphs are denoted by $G_1 \subseteq G_2$, the union of graphs is denoted by $G_1 + G_2$. Inserting vertices (or edges) is denoted by $G + x$ (or $G + xy$, respectively) and deleting edges by $G - xy$. Other notations, which are not explicitly mentioned can be found in Diestel [8].

Definition 1.1. Let $G = (V, E)$ and $m \in \mathbb{N}$. The (open) $m$-neighborhood of $x \in V$ is given by

$$p_m(x : G) = \{y \in V \mid \exists \ xy\text{-path of length } m \text{ in } G\}.$$

If the context to $G$ is clear we write $p_m(x)$ for short.

Note that $p_m$ is symmetric for undirected graphs: $y \in p_m(x) \iff x \in p_m(y)$. For any vertex holds $v \notin p_m(v)$, because a path having distinct ends is required for $p_m, m \geq 1$. The distance of vertices $x$ and $y \in p_m(x)$ is at most $m$.

Using this definition the $m$-step graph is an intuitive way of describing $p_m(v)$ for any $v \in V$.

Definition 1.2. If $G = (V, E)$ is a graph, its $m$-step graph $N_m(G) = (V, E_m)$ is given by

$$E_m = \{xy \mid y \in p_m(x)\}.$$
1 Introduction and Basic Definitions

The trivial cases of definition 1.2 are the following:

- The 1-step graph of $G$ is $N_1(G) = G$ itself, because paths of length 1 in $G$ are given exactly by its edges $E(G)$.
- For $m \geq |V|$ the $m$-step graph of $G$ has no edges, because there is no path of length $m$ with $|V|$ vertices.

Therefore I will consider only $m$-step graphs with $m \geq 2$ and $|V| > m$ avoiding excessive case distinctions. Figure 1 describes a basic example of constructing $m$-step graphs. An elementary result used for constructions of $m$-step graph is given by the following proposition.

**Proposition 1.3.** Let $G$ be a simple graph. Then

$$\forall H \subseteq G : N_m(H) \subseteq N_m(G).$$

**Proof.** Let $H \subseteq G$ be an arbitrary subgraph of $G$. Since $V(N_m(H)) = V(H) \subseteq V(G) = V(N_m(G))$, it follows $V(N_m(H)) \subseteq V(N_m(G))$. Now let $xy \in E(N_m(H))$ be arbitrarily chosen, i.e. $y \in p_m(x : H)$. It follows $y \in p_m(x : G)$, because paths in $H$ are also paths in $G$. Therefore $xy \in E(N_m(G))$.

\[\square\]
2 Literature and Overview

In this section I will describe some topical work using definitions similar to the $m$-step graphs given in the introduction. Afterwards I will describe the competition and embedding number of graphs, which are hard to determine, even for restricted graph classes. After that I will give an overview on the structural properties of $m$-step graphs, which are investigated in this thesis.

2.1 Neighborhood and Competition Graphs

The competition graph $C(D)$ of a directed graph $D$ is a simple graph constructed over the same vertex set of $D$ and having edges $xy \in E(V(D))$ if and only if there exists a vertex $v$ such that $(x, v)$ and $(y, v)$ are arcs in $D$. The term competition graph was introduced by Cohen [6] in 1968 and caused a lot of further research on this topic.

The competition graph of an undirected graph has a handful of equivalent names. In fact, the definition of the 2-step graph $N_2(G)$ is one of those names; it is obtained by replacing the arcs $(x, y)$ and $(y, x)$ in a symmetric digraph by the edge $xy$ or vice versa. Another equivalent definition is that of the neighborhood graph $N(G) = N_2(G)$.

In his Bachelor thesis Pfützenreuter [17] investigated structural properties of neighborhood graphs. Moreover, there have been several interesting studies concerning neighborhood graphs: In 1995 Lundgren et al. [13] characterized graphs which have neighborhood graphs, that are interval or unit interval. Furthermore Lundgren, Merz and Rasmussen [14] investigated the chromatic numbers of competition graphs. Competition graphs of strongly connected and hamiltonian digraphs have been investigated by Fraughnaugh et al. [9] in 1995. Schiermeyer, Sonntag and Teichert [18] investigated the hamiltonicity of neighborhood graphs in 2009. Another generalization was introduced and investigated by Sonntag and Teichert [19], [20], [21] using hypergraphs. The competition hypergraph $C\mathcal{H}(D)$ of a digraph $D$ is defined on the same vertex set $V(D)$ and $e \subseteq V(D)$ is an edge if and only if $|e| \geq 2$ and there is a vertex $v \in V(D)$, such that $e = \{w \in V(D) \mid (w, v) \in A(D)\}$.

When dealing with $m$-step graphs one might come across the definition of the power of a graph. The $k$-th power $G^k$ of a graph is defined on the same vertex set having edges $xy \in E(G^k)$ if and only if their distance is at most $k$, that is $d_G(x, y) \leq k$. Another notion is
2 Literature and Overview

2.2 Embedding and Competition Number

$G^{(k)}$, which describes a graph on the same vertex set having edges $xy \in E(G^{(k)})$ if and only if their distance is exactly $k$. However, in general neither $G^{k}$ nor $G^{(k)}$ are equivalent to $m$-step graphs.

2.2 Embedding and Competition Number

Not all graphs are competition or neighborhood graphs. This will also be discussed in Section 4. However, it is possible to obtain from a graph a competition graph by adding isolated vertices. The least number of isolated vertices needed for this procedure is called the competition number. Similarly every graph $G$ can be embedded in an $m$-step graph $N_m(G')$ as an induced subgraph. The least number of vertices for such a graph $G'$ is called the embedding number.

The embedding number was investigated by Boland, Brigham and Dutton in [2] and [3], based on the introduction of open neighborhood graphs by Acharya and Vartak [1].

Similarly to $m$-step graphs there is a generalization for competition graphs called $m$-step competition graph introduced by Cho, Kim and Nam [5] in 2000. The $m$-step competition graph of a digraph $D$ is defined on the same vertex set and has edges $xy$ if $x$ and $y$ have a common $m$-step pray, that is a vertex $v$ with directed paths of length $m$ from $x$ to $v$ and from $y$ to $v$. They also introduced the $m$-step competition number. Further work on this definition was done by Helleloid [10] in 2004 investigating connected triangle-free $m$-step competition graphs, by Ho [11] in 2005 introducing same-step and any-step competition graphs and by Zhao and Chang in 2009 examining the $m$-step competition number of paths and cycles.

Determining the competition number appears to be a difficult problem: In 1971 Stephen A. Cook [7] published his paper on the concept of NP-completeness. Based on this Richard M. Karp [12] took 21 well-known problems - for which there were (and still are) no deterministic polynomial algorithms found - and proved their NP-completeness. Using these results James Orlin [16] was able to prove the NP-completeness of determining minimal edge-clique-covers (ECCs) in 1977 by reducing this problem amongst others to Karps chromatic number problem. Robert J. Opsut [15] then showed 1982 that the ECC problem is reducible to computing the competition number of graphs. That means, if there was a deterministic polynomial algorithm for computing the competition number, then the infamous equation $P = NP$ would be solved.
2 Literature and Overview

2.3 Overview

In the following I want to give some detailed examples in Section 3, namely the descriptions of $m$-step graphs of well-known graph classes; paths, cycles, wheels and bipartite graphs. Then I will discuss some basic graph properties in Section 4 or to be more specific, I will answer some questions on how much these graph properties are preserved by the $m$-step function. As a first step in this, injectivity and surjectivity of the $m$-step function will be discussed. After that the minimum degree is a perfect example on how a graph property can be preserved by the $m$-step function. Two more of such interesting properties are connectivity and hamiltonicity, which got their own chapters 5 and 6. Finally I will have some conclusions, summaries and open problems in Section 7.
3 Particular Graph Classes

In this chapter I will describe particular $m$-step graphs, namely the $m$-step graphs of paths, cycles, wheels and bipartite graphs. Examining these graph classes will give us a basic idea on how to work with $m$-step graphs, so that we can rely on these results in the further chapters. Considering the complete graph $K_n$ with $n$ vertices, for example, its $m$-step graph is still $K_n$, respecting the condition $2 \leq m < n$ given in the introduction. Therefore by Proposition 1.3 we can conclude, for example, that any supergraph of $K_n$ has again $K_n$ in its $m$-step graph.

3.1 Paths

$P_n$ is a path of length $n - 1$ with $n$ vertices.

Proposition 3.1. Let $d \in [0, m - 1]$ with $d \equiv n \mod m$. The $m$-step graph $N_m(P_n)$ consists of $m$ paths; $d$ of those paths have $\left\lceil \frac{n}{m} \right\rceil$ vertices, the other paths have $\left\lfloor \frac{n}{m} \right\rfloor$ vertices, i.e.

$$N_m(P_n) = d \cdot P_{\left\lceil \frac{n}{m} \right\rceil} + (m - d) \cdot P_{\left\lfloor \frac{n}{m} \right\rfloor},$$

or by substitution $n = m \cdot k + d$ for any $k \in \mathbb{N}$ and $d \in [0, m - 1]$ this is

$$N_m(P_{m \cdot k + d}) = d \cdot P_{k + 1} + (m - d) \cdot P_k.$$

![Diagram](image)

Figure 2: The 5-step graph $N_5(P_{5k+2})$ consists of five paths.
Proof. Let \( n = m \cdot k + d \) and \( P_n = v_0 \ldots v_{m-1} \ldots v_{n-1} \). Since there are at least \( m \) vertices there is a partition of \( V \) containing \( m \) subsets \([v_0], \ldots, [v_{m-1}]\) with

\[
[v_i] := \{ v_k \in V \mid k \equiv i \mod m \}.
\]

The induced subgraphs in \( N_m(P_n) \) having vertices \([v_i]\) are paths.

- There are two possibilities for the number of vertices, which is caused by

\[
[v_i] = \begin{cases} 
 v_i, v_{i+m}, v_{i+2m}, \ldots, v_{i+km} & \text{if } 0 \leq i < d \\
 v_i, v_{i+m}, v_{i+2m}, \ldots, v_{i+(k-1)m} & \text{if } d \leq i < m
\end{cases}
\]

Therefore we obtain \(|[v_i]| = \left\lfloor \frac{n-m}{m} \right\rfloor\), that is

\[
|[v_i]| = \begin{cases} 
 k + 1 = \left\lfloor \frac{n}{m} \right\rfloor + 1 & \text{if } 0 \leq i < d \\
 k = \left\lfloor \frac{n}{m} \right\rfloor & \text{if } d \leq i < m
\end{cases}
\]

However, since \( d = 0 \) always follows the second case, we can rewrite the first case by using \( \left\lceil \frac{n}{m} \right\rceil \) instead of \( \left\lfloor \frac{n}{m} \right\rfloor + 1 \).

- The edges induced are along the path \( v_iv_{i+m}v_{i+2m} \ldots, v_{i+km} \).

- In addition these paths are not interconnected, because there can be no path of length \( m \) from \( v_i \) to \( v_j \) in \( P_{m \cdot k + d} \) with \( i \not\equiv j \mod m \).

 Altogether we obtain \( d \) paths each with \( \left\lceil \frac{n}{m} \right\rceil \) vertices and \( m - d \) paths each with \( \left\lfloor \frac{n}{m} \right\rfloor \) vertices.

\[\square\]

### 3.2 Cycles

\( C_n \) is a cycle with \( n \) vertices, say \( V(C_n) = \{v_0, \ldots, v_{n-1}\}, v_iv_{i+1} \in E(C_n) \) and indices are taken modulo \( n \). For convenience \( C_1 \) denotes a single vertex and \( C_2 \) denotes two connected vertices instead of a real cycle.

**Proposition 3.2.** Let \( g = \gcd(m, n) \). The \( m \)-step graph of \( C_n \) consists of \( g \) cycles of equal length,

\[
N_m(C_n) = g \cdot C_g.
\]
Proof. There is a partition of $V(C_n)$ containing $g$ subsets $\{[v_0], \ldots, [v_g]\}$ with

$$[v_i] := \{v_k \mid k \equiv i \mod g\} \subseteq V.$$

The induced subgraphs in $N_m(C_n)$ having vertices $[v_i]$ are cycles.

- The number of vertices $|[v_i]|$ is the least $k \in \mathbb{N}$ such that

$$i + k \cdot m \equiv i \mod n.$$

By subtracting $i$ on both sides and dividing by $g$ we obtain

$$k \cdot \frac{m}{g} \equiv \frac{n}{g}.$$

Because $\frac{m}{g}$ and $\frac{n}{g}$ are coprime, the least such $k$ is exactly $\frac{n}{g}$.

- The edges induced are along the cycle $v_iv_i+m v_i+2m \ldots v_i+\frac{n}{2}m$ with $v_i+\frac{n}{2}m = v_i$ (indices taken modulo $n$).

- Let $[v_i]$ and $[v_j]$ be any two distinct sets of vertices. The cycles are not interconnected.

This is proven by contradiction. If there was an edge $\{v_{i+a \cdot g}, v_{j+b \cdot g}\} \in N_m(C_n)$ ($a, b \in \mathbb{N}$) we would obtain $i + a \cdot g - (j + b \cdot g) \equiv 0 \mod m$ which means $i - j \equiv 0 \mod g$ and thus $[v_i] = [v_j]$.

Altogether we obtain $g$ cycles each with $|[v_i]| = \frac{n}{g}$ vertices in $N_m(C_n)$. □
3.3 Wheels

A wheel $W_n$ is a graph with one center vertex connected to each vertex of a cycle of $n$ vertices. Because of this notation $n = |V| - 1$ and $m \leq n$.

**Proposition 3.3.** The $m$-step graph of a wheel $W_n$ is the complete graph $K_{n+1}$.

$$N_m(W_n) = K_n.$$ 

**Proof.** Let $V(W_n) = \{v_0, v_1, \ldots, v_n\}$ with center vertex $v_0$ and circle $v_1, \ldots, v_n$. It is sufficient to show, that $v_1$ has paths of length $m$ to each other vertex. Consider the following three cases showing $v_1v_j \in E(N_m(W_n))$ for $i = 0, 2 \leq i < m$ or $m \leq i \leq n$.

\begin{itemize}
  \item Let $i = 0$. Then $v_1 \ldots v_m v_0$ is a path of length $m$ in $W_n$.
  \item Let $2 \leq i < m$. Then $v_1 \ldots v_{i-1} v_0 v_m v_{m-1} \ldots v_i$ is a path of length $m$ in $W_n$.
  \item Let $m \leq i \leq n$. Then $v_1 \ldots v_{m-1} v_0 v_i$ is a path of length $m$ in $W_n$.
\end{itemize}

Therefore $p_m(v_1) = V \setminus \{v_1\}$. Because of the symmetry in a wheel, it follows $p_m(v_i) = V \setminus \{v_i\}$ for $1 \leq i \leq n$. And by the symmetry of $p_m$, from $v_0 \in p_m(v_i)$ for $v_i \in V \setminus \{v_0\}$ it follows $p_m(v_0) = V \setminus \{v_0\}$. 

\[\square\]
3.4 Complete Bipartite Graphs

For a bipartite graph \( G = (A \cup B, E) \) with \( A \cap B = \emptyset \) let \( a = |A|, b = |B| \). Without loss of generalization assume \( a \leq b \). The complete bipartite graph \( K_{a,b} \) is a bipartite graph with all possible edges \( E = \{a_i b_j \mid a_i \in A \land b_j \in B\} \). The graph without edges having \( n \) vertices is denoted by \( I_n \).

**Proposition 3.4.** The \( m \)-step graph of the complete bipartite graph is

\[
N_m(K_{a,b}) = \begin{cases} 
K_{a,b} & \text{if } m \text{ is odd and } m < 2a \\
K_a + K_b & \text{if } m \text{ is even and } m < 2a \\
I_a + K_b & \text{if } m = 2a \text{ and } a < b \\
I_{a+b} & \text{otherwise}
\end{cases}
\]

**Proof.** Since any path of length \( m \) in \( K_{a,b} \) is alternating on \( A \) and \( B \) it can be written in exactly one of the following three notations:

- \( P_{AA} = a_0 b_1 a_2 b_3 \ldots b_{m-1} a_m \) if \( m \) is even
- \( P_{AB} = a_0 b_1 a_2 b_3 \ldots a_{m-1} b_m \) if \( m \) is odd
- \( P_{BB} = b_0 a_1 b_2 a_3 \ldots a_{m-1} b_m \) if \( m \) is even

<table>
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<tr>
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<th>( A )</th>
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<tr>
<td>( P_{AA} )</td>
<td>( \frac{m}{2} + 1 )</td>
<td>( \frac{m}{2} )</td>
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<tr>
<td>( P_{AB} )</td>
<td>( \frac{m+1}{2} )</td>
<td>( \frac{m+1}{2} )</td>
</tr>
<tr>
<td>( P_{BB} )</td>
<td>( \frac{m}{2} )</td>
<td>( \frac{m}{2} + 1 )</td>
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Table 1: Number of vertices of \( A \) and \( B \) traversed by paths \( P_{AA}, P_{AB} \) and \( P_{BB} \).

Table 1 describes the number of vertices of \( A \) and \( B \) traversed by each path \( P_{AA}, P_{AB} \) and \( P_{BB} \). Let \( a_0, a_m \in A \) and \( b_0, b_m \in B \) be arbitrarily chosen vertices. The following existence propositions are true, because \( b \geq a \) and \( K_{a,b} \) is complete.
3.4 Complete Bipartite Graphs

- $P_{AA}$ exists if and only if $a \geq \frac{m}{2} + 1$, i.e. $m \leq 2a - 2$ and $m$ is even.

- $P_{AB}$ exists if and only if $a \geq \frac{m+1}{2}$, i.e. $m \leq 2a - 1$ and $m$ is odd.

- $P_{BB}$ exists if and only if $a \geq \frac{m}{2}$ and $b > a$, i.e. $m \leq 2a$ and $b > a$ and $m$ is even.

For even $m$ the $m$-step graph $N_m(K_{a,b})$ is induced by paths of type $P_{AA}$ and $P_{BB}$. For odd $m$ the $m$-step graph $N_m(K_{a,b})$ is induced only by paths of type $P_{AB}$. Therefore for even $m$ we obtain the union of $K_a$ and $K_b$; for odd $m$ we obtain again the bipartite graph $K_{a,b}$. Only for $m = 2a$ and $b > a$ the paths $P_{BB}$ do exist while paths $P_{AA}$ do not exist; thus we obtain in this case the union of $I_a$ and $K_b$.

□

A star is a graph with one center vertex and $n$ additional vertices connected to its center, thus a star is $K_{1,n}$ and

$$N_m(K_{1,n}) = \begin{cases} 
K_{1,n} & \text{if } m = 1 \text{ (trivial)}, \\
K_n + I_1 & \text{if } m = 2, \\
I_1 + n & \text{otherwise}. 
\end{cases}$$
4 Basic Results for Arbitrary Graphs

In this chapter I will present some basic results on the structure of \(m\)-step graphs. After investigating injectivity and surjectivity of the \(m\)-step function I will answer some other questions similar to that of surjectivity. Then we will discuss a lower bound for the minimum degree of an \(m\)-step graph. After that I will finish this section by investigating some isomorphism problems, that ask for characterizations of graphs \(G\) such that the equations \(N_m(G) = K_n\), \(N_m(G) = G\) or \(N_m(G) = \overline{G}\) are fulfilled.

There are two elemental questions concerning \(m\)-step graphs:

- If two graphs \(G_1\) and \(G_2\) have the same \(m\)-step graph \(N_m(G_1) = N_m(G_2)\), does that imply \(G_1 = G_2\)?

- Is any graph \(G \in \mathcal{G}(V)\) an \(m\)-step graph? That is, for any graph \(G\) is there another \(G' \in \mathcal{G}(V)\) such that \(N_m(G') = G\)?

We can define the \(m\)-step function as a function mapping from simple to simple graphs, i.e.

\[
N_m : \mathcal{G}(V) \to \mathcal{G}(V), \quad N_m : (V, E) \mapsto (V, \{xy \mid y \in p_m(x)\}).
\]

With that the above questions ask for injectivity and surjectivity of this \(m\)-step function.

**Proposition 4.1.** Let \(V\) be any (finite) vertex set and \(m \in \mathbb{N}\) with \(2 \leq m < |V|\). Then the function \(N_m : \mathcal{G}(V) \to \mathcal{G}(V)\) is neither injective nor surjective.

**Proof.** Since \(N_m(K_2) = N_m(I_2) = I_2\) the function \(N_m\) is not injective. Furthermore the domain and codomain are equal and finite, therefore the range of \(N_m\) has less elements than its codomain and thus \(N_m\) is not surjective.

\[\square\]

The restriction to finite graphs makes this proof easier, however, as we will realize later, there are graphs that are not \(m\)-step graphs no matter how many vertices (or edges) a graph is allowed to have.

Concerning the trivial cases of \(m\)-step graphs as described in the introduction, we can complete the above proposition by the following.

- If \(m = 1\), then \(N_m(G) = G\); thus \(N_m\) is bijective.
• Otherwise if \( m \geq |V| \), then \( N_m(G) \) is empty. Therefore \( N_m \) is bijective for \( |V| \leq 1 \) and neither injective nor surjective for \( |V| \geq 2 \).

In order to develop a better understanding we will weaken the condition of surjectivity and examine the results. Let \( G \) be an arbitrary graph in \( \mathcal{G}(V) \). Surjectivity asks for a graph \( G' \in \mathcal{G}(V) \) such that \( N_m(G') = G \). By adding \( t \) vertices to \( G' \) we have \( G' \in \mathcal{G}(V \cup \{v_1, \ldots, v_t\}) \). However, that makes \( N_m(G') \neq G \) in any case, because they are defined on different vertex sets. That is why we should ask for the following questions.

1. Is there a graph \( G' \) such that \( N_m(G') \) contains only \( G \) and \( t \) isolated vertices?
2. Is there a graph \( G' \) such that \( N_m(G') \) contains \( G \) as a component?
3. Is there a graph \( G' \) such that \( N_m(G') \) contains \( G \) as an induced subgraph?

The second and thus the third question as well will be answered positively by the following proposition.

**Proposition 4.2.** For any graph \( G \in \mathcal{G}(V) \) there is a \( t \in \mathbb{N}_0 \) such that there exists a graph \( G' \in \mathcal{G}(V \cup \{v_1, \ldots, v_t\}) \) with \( N_m(G') \) containing \( G \) as a component.

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**Figure 5:** Subdividing edges such that \( G \) is a component of \( N_m(G') \) (\( e_i = v_a v_b, a < b \)).
4 Basic Results for Arbitrary Graphs

Proof. Let \( G = (V, E) \) be a simple graph with vertices \( V = \{v_1, \ldots, v_n\} \) and edges \( E = \{e_1, \ldots, e_k\} \). Then \( G' = (V', E') \) is constructed by subdividing each edge in \( G \) into \( m \) edges.

\[
V' = V \cup \{v^i_j \mid i = 1, \ldots, k, j = 1, \ldots, m-1\}
\]
\[
E' = \{v^i_jv^i_{j+1} \mid i = 1, \ldots, k, j = 1, \ldots, m-2\}
\]
\[
\cup \{v_av^i_1v^m_1v_b \mid v_av_b = e_i, a < b, i = 1, \ldots, k\}
\]

Now \( N_m(G') \) contains \( G \) as an induced subgraph: By construction the vertex set \( V \) is a subset of \( V' \). Let \( e_i = v_av_b \in E \) with \( a < b \) be an arbitrary edge in \( G \). Since \( G' \) contains a path of length \( m \) from \( v_a \) to \( v_b \), namely \( v_av^1_1v^2_1\ldots v^m_1v_b \), this edge \( e_i \) is in \( E(N_m(G')) \) as well. In addition, any path of length \( m \) in \( G' \) with end vertices in \( V(G) \) by construction has a corresponding edge in \( E(G) \). Therefore \( G \) is an induced subgraph of \( N_m(G') \).

It is left to prove, that \( G \) is a component of \( N_m(G) \), which means, that the subgraph is not connected to any vertex in \( V' \setminus V = \{v^i_j \mid i = 1, \ldots, k, j = 1, \ldots, m-1\} \). However, by construction any path of length \( m \) starting in \( v_a \in V(G) \) ends in \( v_b \in V(G) \). Thus because of symmetry, there can be no path from \( v_a \) to \( v^i_1 \in V(G) \) of length \( m \), and there is no edge \( v_av^i_1 \in N_m(G') \).

Furthermore, if \( G \) is a component of \( N_m(G') \) then it is also an induced subgraph, thus positively answering the third question. In that case the least number \( |V(G')| = |V(G)| + t \) is called the embedding number of \( G \). Determining the embedding number seems not to be an easy problem. Boland, Brigham and Dutton have done research on this for neighborhood graphs \( N_2 \) in [2] and [3]. However, the NP-completeness has not been proven yet, and the embedding number of \( m \)-step graphs \( N_m(G), m \geq 3 \) was not investigated yet.

The first question however has a negative answer. For example let \( m = 2 \), then \( G = P_2 = acb \) plus any number of isolated vertices is not a neighborhood graph. For those two edges \( ac \) and \( bc \) there must be two vertices \( v_1, v_2 \) adjacent to the ends of these edges. If they are identical \( v_1 = v_2 \) then there is an edge \( ab \in E(N_2(G)) \) induced by the path \( av_1b \). This yields a triangle instead of a path. If the vertices are not identical \( v_1 \neq v_2 \), then they have a common neighbor \( c \) and thus are connected in the neighborhood graph. For arbitrary \( m \geq 3 \) the graph \( G' \) with \( N_m(G') = P_2 + I_t \) can be constructed as shown in Figure 6.
Basic Results for Arbitrary Graphs

Figure 6: An $m$-step graph consisting of $P_2 = acb$ and isolated vertices for $m \geq 3$.

However, if $G = P_3$ there is no graph $G'$ with $N_m(G') = P_3 + I_r$.

**Proposition 4.3.** Let $m \geq 2$ and $t \in \mathbb{N}_0$. Then there is no graph $G'$ such that $N_m(G') = P_3 + I_r$.

**Proof.** Let $P_3 = acbd$ a path of length three. The subpath $acb$ must be a fork in $G'$ as can be seen in Figure ??: Let $ac$ be induced by the path $a = a_0a_1...a_m = c$ and $bc$ induced by the path $b = b_0b_1...b_m = c$. It follows, that there must be a minimal $i < \frac{m}{2}$ with $a_i = b_i$; otherwise this would yield edges in $N_m(G')$ that are not in $acbd$. Now there must be a path of length $m$ in $G'$ inducing the edge $bd$. However, by a case distinction on where this path must diverge from the other paths, in any case there is a fourth edge or a triangle induced. Therefore we obtain contradictions such that there is no $G'$ with $N_m(G') = P_3 + I_r$.

\[\square\]

Figure 7: Any $bd$-path in $G$ implies edges in $N_m(G')$, that are not in $acbd$. 
4.1 Minimum Degree

A lower bound for the minimum degree of $m$-step graphs is given by

**Theorem 4.4.** Let $G = (V, E)$ be a graph with minimum degree $\delta(G)$ and $m \in \mathbb{N}$ with $2 \leq m \leq \delta(G)$. Then we obtain for the $m$-step graph $N_m(G)$

$$\delta(N_m(G)) \geq \delta(G) - 1. \quad (1)$$

**Proof.** Without loss of generality assume that $G$ is connected (otherwise the following considerations can be made separately for each component). We have $|V| > \delta(G)$ and because of $m \leq \delta(G)$ there is a path $P = v_1 \ldots v_m$ of length $m - 1$ in $G$ for an arbitrarily chosen start vertex $v_1 \in V$. In the following we show the existence of $\delta(G) - 1$ paths of length $m$ from $v_1$ to pairwise distinct end vertices, which proves (1).

(a) Let $j = ||v_iv_m \in E \mid i \in \{1, \ldots, m - 2\}||$, that is the number of edges from any vertex $v_i$ in the path (except from $v_{m-1}$) to the end vertex $v_m$. Then $v_m$ has at least $\delta(G) - (j + 1)$ neighbors $a_1, \ldots, a_{\delta(G) - j - 1} \notin V(P)$. Now the path $Pa_i$ from $v_1$ to $a_i$ is of length $m$ in $G$, thus $v_1a_i \in E(N_m(G))$ for $i = 1, \ldots, \delta(G) - j - 1$ (see Figure (8)).

![Figure 8: Path $v_1 \ldots v_m$, neighbors of $v_m$ and $j = 2$ edges from $v_m$ back to the path.](image)

(b) Now consider the $j$ vertices $v_{i_1}, \ldots, v_{i_j} \in V(P)$ with $i_t \in \{2, \ldots, m - 1\}$, $v_{i_t-1}v_m \in E$ and $t = 1, \ldots, j$. Because of $\delta(G) \geq m$ there are (not necessarily distinct) vertices $w_t \notin V(P)$ with $w_tv_{i_t+1} \in E$ and $t = 1, \ldots, j$. For each $t \in \{1, \ldots, j\}$ we distinguish three cases (see Figure (9)).

(i) Let $b_t := v_{i_t+2}$ and $w_tv_m \in E$ (that is $w_t = a_t, i \in \{1, \ldots, \delta(G) - j - 1\}$). Then the path $P_1 = v_1 \ldots v_{i_t}v_{i_t+1}w_tv_m \ldots b_t$ has length $m$, and thus $v_1b_t \in E(N_m(G))$.

(ii) Let $b_t := w_t$ and $w_tv_m \notin E$. If there is no other vertex $v_{i_t}$ with $i_t > i_t$ and $\{v_{i_t+1}, w_t\} \in E$ then the path $P_2 = v_1 \ldots v_{i_t}v_m \ldots v_{i_t+1}b_t$ has length $m$ and $\{v_1, b_t\} \in E(N_m(G))$. Note that this case appears at most once for each vertex $w_t$. 
4.1 Minimum Degree

(iii) Otherwise there is a vertex \(v_{i_r}\) with \(i_r > i_t\) and \({v_{i_t + 1}, w_t}\) \(\in E\). Let \(b_t := v_{i_t + 2}\). Then the path \(P_3 = v_1 \ldots v_{i_t} v_{i_t + 1} w_t v_{i_t + 2} \ldots v_m v_{i_r} \ldots b_t\) is of length \(m\) and \(v_1 b_t \in E(N_m(G))\).

Summarizing the results we obtain the vertices \(a_1, \ldots, a_\delta(G) - j - 1\) and \(b_1, \ldots, b_j\). In cases (i) and (iii) \(b_t = v_{i_t + 2}\) and in case (ii) \(b_t = w_t\). Furthermore these vertices are pairwise distinct and we obtain \(\delta(G) - 1\) edges \({v_1, a_i}, {v_1, b_i} \in E(N_m(G))\), this completes the proof.

\[\square\]

This lower bound for the minimum degree is sharp.

- For \(K_n\) we have \(\delta(K_n) = n - 1\) and \(\delta(N_m(K_n)) = 0\) for any \(m \geq n > \delta(K_n)\).

- There are graphs with \(\delta(N_m(G)) = \delta(G) - 1\). Let \(m \in \mathbb{N}\) and \(m \geq 2\). Graph \(G = (V, E)\) is the union of a tree and a complete bipartite graph; the tree has the root vertex \(v\) and all leaves are at depth \(m - 1\), the inner nodes have degree of \(m\) and all the leaf vertices are connected to the vertices \({y_1, \ldots, y_{m-1}}\). See Figure 10 for an example. To be precise, the construction is formally given by

\[
V = \bigcup_{i=0}^{m} L_i, \quad L_i = \begin{cases} 
\{v\} & \text{if } i = 0 \\
\{v_p \mid p \in [1, m] \times [1, m - 1]^{i-1}\} & \text{if } 0 < i < m \\
\{y_k \mid k \in [1, m - 1]\} & \text{if } i = m 
\end{cases}
\]
and the set of edges $E$ is defined by the following four conditions

(i) $\{vv_1, \ldots, vv_m\} \subseteq E$,

(ii) $\forall i \in [1, m-2] \forall w \in L_i : w = v_{p_1, \ldots, p_i} \rightarrow \{wv_{p_1, \ldots, p_i, 1}, \ldots, wv_{p_1, \ldots, p_i, m-1}\} \subseteq E$,

(iii) $\forall w \in L_{m-1} : \{wy_1, \ldots, wy_{m-1}\} \subseteq E$,

(iv) there is no other edge in $E$ than those given in (i), (ii) and (iii).

Figure 10: Example for $m = \delta (G) = 4$ and $\delta (N_m(G)) = \delta (G) - 1$.

The graph $G$ has the minimum degree $\delta (G) = m$. Because of Theorem 4.4 the minimum degree of its $m$-step graph is $\delta (N_m(G)) \geq \delta (G) - 1$. Considering the start vertex $v \in G$ we notice that every path of length $m$ is of the type $vv_{p_1, p_3, p_2, \ldots, p_m, 1} y_k$. Therefore $N_m(v) = \{y_1, \ldots, y_{m-1}\}$ and thus $\delta (N_m(G)) = m - 1$.

### 4.2 Isomorphism Problems

Brigham and Dutton [4] gave characterizations for graphs $G$ such that $N_2(G) \cong K_n$ and $N_2(G) \cong G$. Furthermore they described new results on the more difficult problem $N_2(G) \cong \overline{G}$. With respect to $m$-step graphs, these problems can be generalized by the following equations:

- $N_m(G) \cong K_n$,
- $N_m(G) \cong G$ or
- $N_m(G) \cong \overline{G}$.

Let us first consider the equation $N_m(G) \cong K_n$ (remember that $|V| = n > m \geq 2$).
Proposition 4.5. Necessary conditions for $N_m(G) = K_n$ are:

(i) $G$ is connected,

(ii) diameter $d(G) \leq m$,

(iii) if $v \in V(G)$ is a cut vertex separating $A, B$, then $|V(A)| > m$ and $|V(B)| > m$,

(iv) there is no bridge in $G$,

(v) each $e \in E(G)$ is part of a cycle of length $m + 1$.

Proof. If $d(G) > m$, then there are two vertices $x, y \in V(G)$, such that there is no $xy$-path of length $m$, thus $xy \notin E(N_m(G))$. Therefore (ii) is necessary, which implies, that (i) is necessary too. Condition (iii) is necessary, because any $xy$-path in $G$ requires at least $m + 1$ vertices and can not visit a cut vertex twice. Any edge $e = xy \in E(G)$ requires an $xy$-path of length $m$ in $G$, otherwise $e \notin E(N_m(G))$. Therefore $e$ is part of a cycle of length $m + 1$, and (v) is necessary, which implies, that (iv) is necessary too.

Obviously these conditions are not sufficient for $m \geq 3$; $C_{m+1}$ for example fulfills these necessary conditions, but $N_m(C_{m+1}) \cong C_{m+1} \neq K_{m+1}$.

Let us now consider the equation $N_m(G) \cong G$. For $m = 2$ Brigham and Dutton have shown, that every component of $G$ is a complete graph on other than two vertices or an odd cycle. However, for $m \geq 3$ we obtain only sufficient but not necessary conditions by generalizing these conditions. If $n > m$ then $N_n(K_n) = K_n$, and if also $\gcd(m, n) = 1$ then $N_n(C_n) \cong C_n$, but another example is given in Figure 11 showing a spiked cycle isomorphic to its 3-step graph. Similarly for any odd $m$ there is a spiked cycle $G = C_{2m-2} + x + xv_0$, such that $N_m(G) \cong G$. Finding elegant conditions for problems with $m \geq 3$ remains an open problem.

![Figure 11: The spiked cycle on five vertices is isomorphic to its 3-step graph.](image-url)
5 Connectivity

A necessary condition for the connectivity of an $m$-step graph $N_m(G)$ is the connectivity of
$G$, as described in the following proposition.

**Proposition 5.1.** If $N_m(G)$ is connected, then $G$ is also connected.

**Proof.** Let $x, y \in V$ be any arbitrary vertices. Since $N_m(G)$ is connected, there is a path $P$
from $x$ to $y$ in $N_m(G)$. An edge of this path is induced by a path of length $m$ in $G$. Thus there
is a walk from $x$ to $y$ in $G$ and $G$ is connected.

Now the other way round is more difficult. Does connectivity of $G$ imply also the connec-
tivity of $N_m(G)$? From Section 3.1 we already know, that a path $P$ is split up into $m$ paths
in $N_m(P)$. Furthermore $N_m(K_{1,n}) = I_{n+1}$ if $m \geq 3$ (see 3.4), thus there is no boundary for
the number of components of an $m$-step graph $N_m(G)$ of a connected graph $G$. However,
one might come to the idea, that if a graph is connected, large enough and contains certain
subgraphs, then perhaps the connectivity of $N_m(G)$ is guaranteed. An example of such a
subgraph is the cycle $C_{m+1}$. Since $N_m(C_{m+1}) = C_{m+1}$ remains connected this cycle induces
connectivity of any supergraph. More general we obtain

**Proposition 5.2.** Let $G$ be connected and $H \subseteq G$ with $N_m(H)$ connected and having more
than one vertex, then $N_m(G)$ is connected.

**Proof.** Let $x, y \in V(G)$ be arbitrarily chosen. It is to prove, that there is an $xy$-path in $N_m(G)$.

(i) If $x, y \in V(H)$ then there is an $xy$-path in $N_m(G)$, because $N_m(H) \subseteq N_m(G)$ (by Proposi-
tion 1.3) and $N_m(H)$ is connected.

(ii) Let $x \notin V(H), y \in V(H)$. Since $G$ is connected, there is a path $P_1$ from $x$ to $v \in V(H)$ in
$G$ such that $\{v\} = V(P) \cap V(H)$ (see Figure 12). The length of $P_1$ is $k \cdot m + d$ with $k \in \mathbb{N}$
and $d \in [0, m - 1]$. Now let $v_m \in V(H)$ such that there is a $vv_m$-path $vv_1 \ldots vv_{m-1}v_m$ of
length $m$ in $H$. Then $xP_1v_1 \ldots v_{m-d}$ is a path in $G$ with a length divisible by $m$. Therefore
there is a path from $x$ to $v_{m-d} \in V(H)$ in $N_m(G)$. By extending this path with an $v_{m-d}y$-
path as in (i) we obtain a walk from $x$ to $y$ in $N_m(G)$ and thus have an $xy$-path in $N_m(G)$.

(iii) Let $x, y \notin V(H)$ and $v \in V(H)$ arbitrarily chosen. By (ii) we obtain an $xy$-path and an
$yy$-path, therefore there exists an $xy$-path in $N_m(G)$. 

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In any case there exists an $xy$-path in $N_m(G)$ for arbitrarily chosen $x, y \in V(G)$, therefore $N_m(G)$ is connected.

\[\square\]

A subgraph $H \subseteq G$ as in above proposition is called minimal, if there is no other subgraph $H' \subset H$ of at least two vertices such that $N_m(H')$ is connected. For $m = 2$ it is not difficult to see, that the only minimal graphs $H$ with $N_2(H)$ connected are odd cycles:

- Odd cycles are connected and contain more than two vertices. Since $N_2$ preserves odd cycles (under isomorphism) Proposition ?? holds.

- Odd cycles are minimal. Proper connected subgraphs of odd cycles are paths, their $m$-step graphs are disconnected by Proposition 3.1.

- In order to show, that there are no minimal subgraphs $H \subseteq G$ other than odd cycles inducing connectivity in $N_2(G)$, assume $G$ does not contain odd cycles. Then $G$ is bipartite and by Proposition 3.4 $N_2(G)$ is disconnected.

By the same reason an odd cycle is required even though not sufficient for even $m \geq 4$. However, for odd $m \geq 3$ a minimal subgraph $H$ with $N_m(H)$ does not require any cycles. To give an example I will show the connectivity of an acyclic graph, a caterpillar graph that is a tree having its leaf vertices within a distance of 1 from a central (longest) path.

Figure 12: Showing the existence of an $xy$-path for Proposition 5.2 (ii).
Proposition 5.3. For any odd \( m \geq 3 \) the \( m \)-step graph \( N_m(G) \) of the following caterpillar graph \( G = (V, E) \) (see Figure 13) is connected (and even contains a hamiltonian path):

\[
V = \{a_i, b_i, c_i, d_i \mid i = 1, \ldots, m - 1\},
\]
\[
E = \{a_ia_{i+1}, b_ib_{i+1}, c_ic_{i+1} \mid i = 1, \ldots, m - 2\}
\]
\[
\cup \{b_id_i \mid i = 1, \ldots, m - 1\}
\]
\[
\cup \{a_{m-1}b_1, b_{m-1}c_1\}.
\]

\( G : \)

```
Figure 13: The caterpillar graph of Proposition 5.3.
```

Proof. For odd \( m \geq 3 \) the \( m \)-step graph \( N_m(G) \) contains the following edges

\[
E(N_m(G)) = \{a_ia_{i+1}, b_ib_{i+1}, c_ic_{i+1} \mid i = 1, \ldots, m - 2\}
\]
\[
\cup \{a_id_i, d_ic_i \mid i = 1, \ldots, m - 1\} \cup \{c_1a_{m-1}, d_md_{m-1}\}.
\]

Therefore \( N_m(G) \) contains a hamiltonian path \( P = b_1P_1a_{m-1}c_1P_2b_{m-1} \) (see Figure 14) with

\[
P_1 = (b_1c_2d_2a_2)(b_3c_4d_4a_4) \ldots (b_{m-2}c_{m-1}d_{m-1}a_{m-1}),
\]
\[
P_2 = (c_1d_1a_1b_2)(c_3d_3a_3b_4) \ldots (c_{m-2}d_{m-2}a_{m-2}b_{m-1}).
\]

Note that the constructed path uses every edge in \( E(N_m(G)) \) except for \( d_1d_{m-1} \). Furthermore this caterpillar graph is minimal regarding the above definition, i.e. for any proper subgraph \( H \subset G (|V(H)| \geq 2) \) the \( m \)-step graph \( N_m(H) \) is disconnected.
Figure 14: Proving the connectivity by finding a Hamiltonian path in $N_m(G)$. 
6 Hamiltonicity

In this chapter I want to generalize some ideas on hamiltonicity concerning neighborhood graphs.

A simple result follows from theorem 4.4 and Dirac’s theorem stating: Any simple graph $G$ on $n \geq 3$ vertices is hamiltonian if each vertex has degree at least $\frac{n}{2}$.

**Corollary 6.1.** Let $G = (V, E)$ be a simple graph, $n = |V| \geq 3$ and $m < n$. If $\delta(G) \geq \frac{n}{2} + 1$ then $N_m(G)$ is hamiltonian.

**Proof.** From $\delta(G) \geq \frac{n}{2} + 1$ theorem 4.4 follows $\delta(N_m(G)) \geq \frac{n}{2}$, which implies a hamiltonian cycle according to Dirac.

In their paper Schiermeyer, Sonntag and Teichert [18] have proven some interesting propositions, answering the question on how $N_2(G)$ does inherit hamiltonicity properties from $G$. Their basic results are

**Proposition 6.2.** Let $G = (V, E)$ be a graph and $N_2(G)$ its neighborhood graph.

(i) If $|V|$ is odd and $G$ is hamiltonian then $N_2(G)$ is hamiltonian.

(ii) If $G$ is nonbipartite and hamiltonian then $N_2(G)$ contains a hamiltonian path.

(iii) If $G$ has an odd spanning spiked cycle then $N_2(G)$ is hamiltonian.

(iv) If $G$ is 1-hamiltonian then $N_2(G)$ is hamiltonian.

The first proposition can easily be generalized and proven; if $\gcd(m, |V|) = 1$ and $G$ is hamiltonian then $N_m(G)$ is hamiltonian, because by Proposition 3.2 a hamiltonian cycle is congruent to a hamiltonian cycle in $N_m(G)$.

With that in mind one might come to the conclusion, that the odd spanning spiked cycle of 6.2 (iii) can be generalized to a spanning spiked cycle of length $n$ with $\gcd(m, n) = 1$. However, this is not the case. A counterexample is the 3-step graph of the spiked cycle as already seen in Figure 11. Instead there are variants of spikes for which the $m$-step graph is hamiltonian:

- A non-cycle vertex connected to two cycle vertices at a given distance (Proposition 6.3).
Hamiltonicity

- Two spikes at a given distance on the cycle (Proposition 6.4).
- Cycles of length $m$ can be appended to cycle vertices (Proposition 6.5).
- Pairs of paths of length $m - 1$ can be appended to one cycle vertex (Proposition 6.6).

**Proposition 6.3.** Let $G = (V, E)$ with $V = \{v_0, \ldots, v_{n-1}, w\}$. Furthermore let $E = \{wv_a, wv_b\} \cup \{v_i v_{i+1} \mid i = 0, \ldots, n-1\}$ (indices are taken modulo $n$) such that $b - a \equiv m - 2 \mod n$ (or $a - b \equiv m - 2 \mod n$), then $N_m(G)$ is hamiltonian.

**Proof.** Without loss of generalization assume $a = 1$ and $b = m - 1$ (otherwise this can be achieved by switching $a$ and $b$ or by rotation ($v_i \rightarrow v_{i+\tau}$, indices taken modulo $n$). From $\gcd(m, n) = 1$ and Proposition 3.2 we obtain a cycle in $N_m(G)$ covering the cycle vertices $v_0, \ldots, v_{n-1}$. To integrate the vertex $w$ in this cycle there are three edges of importance (see Figure 15):

- $wv_0 \in E(N_m(G))$ since there is a path of length $m$ in $G$, namely $wv_{m-1}v_{m-2}\ldots v_1v_0$.
- $wv_m \in E(N_m(G))$ since there is a path of length $m$ in $G$, namely $wv_1v_2\ldots v_{m-1}v_m$.
- $v_0v_m \in E(N_m(G))$ is obtained from the induced cycle of $C_N$.

Now $(V, \{v_i v_{i+m} \mid i = 1, \ldots, n-1\} \cup \{wv_0, wv_m\}) \subseteq N_m(G)$ is a hamiltonian cycle.

\[\square\]

![Figure 15: Replacing $v_0v_m$ (blue) by $v_0wv_m$ (green) yields a hamiltonian cycle.](image)

In case of $m = 2$ the precondition $b - a \equiv m - 2 \mod n$ means $b = a$ and therefore $v_a = v_b$, yielding the familiar spike in Proposition 6.2 (iii).

Another possibility exists by using two spikes at a given distance:
Proposition 6.4. Let $G = (V, E)$ with $V = \{v_0, \ldots, v_{n-1}, x, y\}$. Furthermore let $E = \{xv_a, yv_b, xy\} \cup \{v_i v_{i+1} \mid i = 0, \ldots, n-1\}$ (indices are taken modulo $n$) such that $b - a \equiv m - 2 \mod n$ (or $a - b \equiv m - 2 \mod n$), then $N_m(G)$ is hamiltonian.

Proof. Similar to the proof of the preceding proposition, assume $a = 1$ and $b = m - 1$ without loss of generelization. There are four edges of importance to construct a hamiltonian cycle:

- $v_0y \in E(N_m(G))$, since $v_0v_1 \ldots v_{m-2}v_{m-1}y$ is a path of length $m$ in $G$.
- $v_mx \in E(N_m(G))$, since $xv_1v_2 \ldots v_{m-1}v_m$ is a path of length $m$ in $G$.
- $xy$, since $xv_1v_2 \ldots v_{m-2}v_{m-1}y$ is a path of length $m$ in $G$.
- $v_0v_m \in E(N_m(G))$ is obtained from the induced cycle of $C_N$.

Therefore $(V, \{v_i v_{i+1} \mid i = 1, \ldots, n-1\} \cup \{xv_0, xy, vy_m\}) \subseteq N_m(G)$ is a hamiltonian cycle.

\[\square\]

Figure 16: Replacing $v_0v_m$ (blue) by $v_0xyv_m$ (green) yields a hamiltonian cycle.
Furthermore there can be cycles of length $m$ be appended to a vertex of $C_n$.

**Proposition 6.5.** Let $G = (V, E)$ with $V = \{v_0, \ldots, v_{n-1}, w_1, \ldots, w_{m-1}\}$. Furthermore let $E = \{v_0w_1, w_1w_2, \ldots, w_{m-2}w_{m-1}, w_{m-1}v_0\} \cup \{v_i v_{i+1} \mid i = 0, \ldots, n-1\}$ (indices are taken modulo $n$), then $N_m(G)$ is hamiltonian.

**Proof.** To construct a hamiltonian cycle there are the following edges of importance:

(i) $v_{n-m+1}w_1, \ldots, v_{n-1}w_{m-1} \in E(N_m(G))$ and $w_1v_1, \ldots, w_{m-1}v_{m-1} \in E(N_m(G))$; these edges are interconnecting the vertices from both cycles.

(ii) The edges $v_{n-m+1}v_1, \ldots, v_{n-1}v_{m-1} \in E(N_m(G))$ are induced by the cycle $C_n$.

Therefore by replacing the edges of (ii) by the paths $v_{n-m+1}w_1v_1, \ldots, v_{n-1}w_{m-1}v_{m-1}$ of (i) yields a hamiltonian cycle, namely $(V, \{v_i v_{i+m} \mid i = 0, \ldots, n-m\} \cup \{v_{n-m+i}w_i, w_iv_i \mid i = 1, \ldots, m-1\})$.

□

An example for constructing this hamiltonian cycle is given in Figure 17 with $m = 6$.

![Figure 17: Replacing $v_{n-m+i}v_i$ (blue) by $v_{n-m+i}w_i$ (green), $i = 1, \ldots, m-1$ with $m = 6$](image-url)
Another possibility is described in Figure 18, which is similar to the previous one using two paths of length $m$ instead of a cycle $C_m$.

**Proposition 6.6.** Let $G = (V, E)$ with $V = \{v_0, \ldots, v_{n-1}, x_1, \ldots, x_{m-1}, y_1, \ldots, y_{m-1}\}$. Furthermore let $E = \{v_0x_1, x_1x_2, \ldots, x_{m-2}x_{m-1}, v_0y_1, y_1y_2, \ldots, y_{m-2}y_{m-1}\} \cup \{v_iv_{i+1} \mid i = 0, \ldots, n-1\}$ (indices are taken modulo $n$), then $N_m(G)$ is hamiltonian.

**Proof.** Similar to the previous proof a hamiltonian cycle is constructed by replacing the edges $v_{n-m+1}v_1, \ldots, v_{n-1}v_{m-1} \in E(N_m(G))$ by the paths $v_{n-m+1}x_1y_{m-i}v_1, \ldots, v_{n-1}x_{m-1}y_1v_{m-1}$, namely $(V, \{v_iv_{i+m} \mid i = 0, \ldots, n-m\} \cup \{v_{n-m+i}x_i, x_iy_{m-i}, y_{m-i}v_i \mid i = 1, \ldots, m-1\})$.

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**Figure 18:** Replacing $v_{n-m+i}v_i$ (blue) by $v_{n-m+i}x_iy_{m-i}v_i$ (green), $i = 1, \ldots, m-1$ with $m = 6$
7 Conclusions

The $m$-step graph is an interesting theoretical construct and closely related to the topical research on neighborhood graphs, competition graphs, $m$-step competition graphs and their other generalizations. Describing the $m$-step graphs for particular graph classes is a straightforward procedure. The more interesting question is: How are graph properties preserved by the $m$-step function? The minimum degree was a perfect example; as long as $m \leq \delta(G)$, the minimum degree of the $m$-step graph is at least $\delta(N_m(G)) \geq \delta(G) - 1$. Other interesting graph properties could be the girth and circumference. Furthermore the results for connectivity 5 and hamiltonicity 6 are elemental results, but there are many questions left. The following section describes a small selection of open problems concerning $m$-step graphs.

7.1 Open Problems

- For $m \geq 3$ it was not difficult to show that $P_2$ together with isolated vertices is an $m$-step graph. $P_3$ was shown to be not an $m$-step graph, no matter how many isolated vertices are added. By that one might conjecture that even paths are $m$-step graphs and odd paths are not. For $m = 2$ however it is obvious that no path of length at least two is a neighborhood graph.

- Is there an elegant way of describing the $m$-step graph of a tree? The easy thing about trees is the existence and uniqueness of $xy$-paths for arbitrary $x, y \in V(G)$. However, as we have seen in Section 5 the $m$-step graphs of trees do not necessarily decompose for odd $m$, which makes an elegant description difficult.

- In Section 2.2 we have seen the history of NP-completeness for determining the competition number. However, because the proof of Opsut [15] makes heavily use of directed arcs, it is difficult and perhaps impossible to translate it to the undirected case, i.e. the embedding number.

- The isomorphism problems in Section 4.2 ask for classes of graphs fulfilling the equations $N_m(G) = K_n$, $N_m(G) = G$, $N_m(G) = \overline{G}$. However, no elegant descriptions of the graphs fulfilling these equations are known yet. Another interesting problem could arises when comparing $N_m^2 = N_m(N_m(G))$ with $N_{2m}(G)$, or in general $N^k_m(G) = N_{m-k}(G)$?
References


