University of Lübeck

Institute of Mathematics

Structural Properties of *m*-Step Graphs

Bachelor Thesis

submitted by	René Schönfelder
course of study	Information Science
matriculation number	560906
address	Weberkoppel 4a
	23562 Lübeck
e-mail	reneschoenfelder@web.de
submitted on	November 18, 2009
advisor	PD Dr. Hanns-Martin Teichert

Declaration

I hereby declare that I produced this thesis without external assistance, and that no other than the listed references have been used as sources of information.

Lübeck, November 18, 2009

Contents

1 Introduction and Basic Definitions

The definition of m-step graphs first requires precise definitions of graphs and paths.

Throughout this thesis I will only consider simple graphs; simple in this context means finite, undirected and having neither loops nor multiple edges. Thus a graph $G = (V, E)$ is a pair of disjoint sets $V = V(G)$, the vertices, and $E = E(G)$, the edges; thereby any edge $e \in E$ is a set of two distinct elements $x, y \in V$. An edge $\{x, y\} \in E$ will be written as $xy \in E$. The set of all possible simple graphs over V is denoted by $\mathcal{G}(V) = \{(V, E) \mid \forall e \in E : e \subseteq V \wedge |e| = 2\}$.

Paths are graphs isomorphic to $P_n = (V, E)$ with $n \in \mathbb{N}$ vertices $V = \{v_1, \ldots, v_n\}$ and edges $E = \{v_1v_2, \ldots, v_{n-1}v_n\}$. The length of a path is the number of its edges $|E| = n - 1$. Its end vertices are v_1 and v_n and the path is called a v_1v_n-path. The inner vertices are $v_2, \ldots, v_{n-1}$. A path from v_1 to v_n is often denoted by the sequence of its vertices $v_1v_2\ldots v_n$. The vertices of P (and therefore its edges) are pairwise distinct, otherwise it is called a walk.

Isomorphism is denoted by $G_1 \cong G_2$, subgraphs are denoted by $G_1 \subseteq G_2$, the union of graphs is denoted by G_1+G_2. Inserting vertices (or edges) is denoted by $G+x$ (or $G+xy$, respectively) and deleting edges by $G - xy$. Other notations, which are not explicitly mentioned can be found in Diestel [8].

Definition 1.1. *Let $G = (V, E)$ and $m \in \mathbb{N}$. The (open) m-neighborhood of $x \in V$ is given by*

$$p_m(x : G) = \{y \in V \mid \exists\ xy\text{-path of length } m \text{ in } G\}\,.$$

If the context to G is clear we write $p_m(x)$ for short.

Note that p_m is symmetric for undirected graphs: $y \in p_m(x) \leftrightarrow x \in p_m(y)$. For any vertex holds $v \notin p_m(v)$, because a path having distinct ends is required for $p_m, m \geq 1$. The distance of vertices x and $y \in p_m(x)$ is at most m.

Using this definition the m-step graph is an intuitive way of describing $p_m(v)$ for any $v \in V$.

Definition 1.2. *If $G = (V, E)$ is a graph, its m-step graph $N_m(G) = (V, E_m)$ is given by*

$$E_m = \{xy \mid y \in p_m(x)\}\,.$$

The trivial cases of definition 1.2 are the following:

- The 1-step graph of G is $N_1(G) = G$ itself, because paths of length 1 in G are given exactly by its edges $E(G)$.
- For $m \geq |V|$ the m-step graph of G has no edges, because there is no path of length m with $|V|$ vertices.

Therefore I will consider only m-step graphs with $m \geq 2$ and $|V| > m$ avoiding excessive case distinctions. Figure 1 describes a basic example of constructing m-step graphs. An ele-

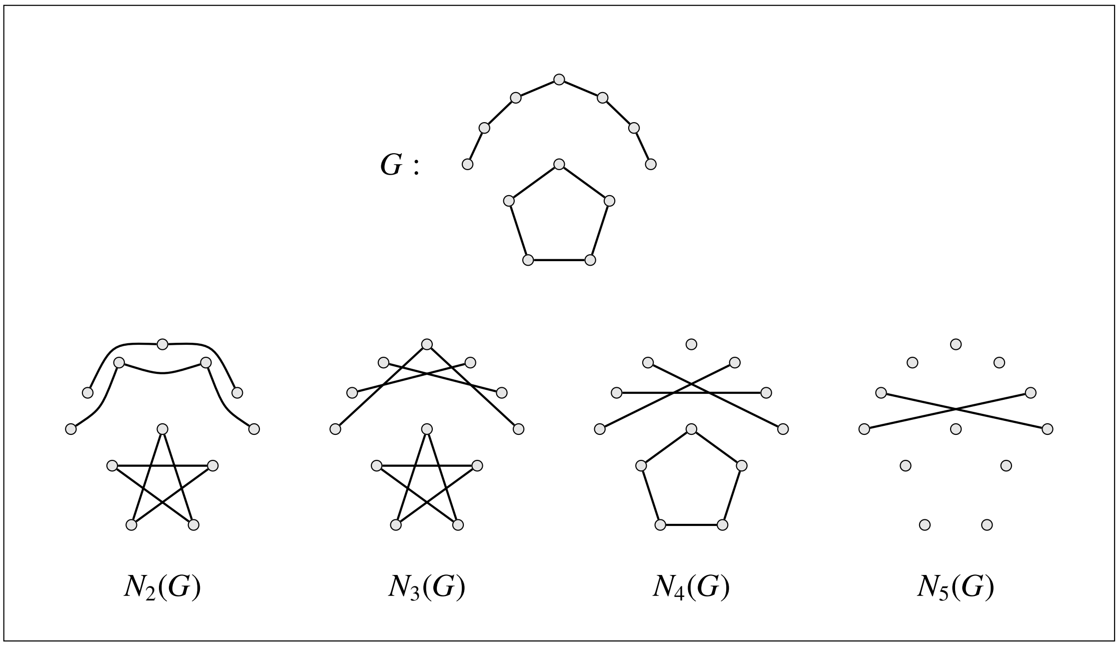

Figure 1: **The 2-, 3-, 4- and 5-step graph of an exemplary graph.**

mentary result used for constructions of m-step graph is given by the following proposition.

Proposition 1.3. *Let G be a simple graph. Then*

$$\forall H \subseteq G : N_m(H) \subseteq N_m(G).$$

Proof. Let $H \subseteq G$ be an arbitrary subgraph of G. Since $V(N_m(H)) = V(H) \subseteq V(G) = V(N_m(G))$, it follows $V(N_m(H)) \subseteq V(N_m(G))$. Now let $xy \in E(N_m(H))$ be arbitrarily chosen, i.e. $y \in p_m(x : H)$. It follows $y \in p_m(x : G)$, because paths in H are also paths in G. Therefore $xy \in E(N_m(G))$.

□

2 Literature and Overview

In this section I will describe some topical work using definitions similar to the m-step graphs given in the introduction. Afterwards I will describe the competition and embedding number of graphs, which are hard to determine, even for restricted graph classes. After that I will give an overview on the structural properties of m-step graphs, which are investigated in this thesis.

2.1 Neighborhood and Competition Graphs

The competition graph $C(D)$ of a directed graph D is a simple graph constructed over the same vertex set of D and having edges $xy \in E(V(D))$ if and only if there exists a vertex v such that (x, v) and (y, v) are arcs in D. The term competition graph was introduced by Cohen [6] in 1968 and caused a lot of further research on this topic.

The competition graph of an undirected graph has a handfull of equivalent names. In fact, the definition of the 2-step graph $N_2(G)$ is one of those names; it is obtained by replacing the arcs (x, y) and (y, x) in a symmetric digraph by the edge xy or vice versa. Another equivalent definition is that of the neighborhood graph $N(G) = N_2(G)$.

In his Bachelor thesis Pfützenreuter [17] investigated structural properties of neighborhood graphs. Moreover, there have been several interesting studies concerning neighborhood graphs: In 1995 Lundgren et al. [13] characterized graphs which have neighborhood graphs, that are interval or unit interval. Furthermore Lundgren, Merz and Rasmussen [14] investigated the chromatic numbers of competition graphs. Competition graphs of strongly connected and hamiltonian digraphs have been investigated by Fraughnaugh et al. [9] in 1995. Schiermeyer, Sonntag and Teichert [18] investigated the hamiltonicity of neighborhood graphs in 2009. Another generalization was introduced and investigated by Sonntag and Teichert [19], [20], [21] using hypergraphs. The competition hypergraph $C\mathcal{H}(D)$ of a digraph D is defined on the same vertex set $V(D)$ and $e \subseteq V(D)$ is an edge if and only if $|e| \geq 2$ and there is a vertex $v \in V(D)$, such that $e = \{w \in V(D) \mid (w, v) \in A(D)\}$.

When dealing with m-step graphs one might come across the definition of the power of a graph. The k-th power G^k of a graph is defined on the same vertex set having edges $xy \in E(G^k)$ if and only if their distance is at most k, that is $d_G(x, y) \leq k$. Another notion is

$G^{(k)}$, which describes a graph on the same vertex set having edges $xy \in E(G^{(k)})$ if and only if their distance is exactly k. However, in general neither G^k nor $G^{(k)}$ are equivalent to m-step graphs.

2.2 Embedding and Competition Number

Not all graphs are competition or neighborhood graphs. This will also be discussed in Section 4. However, it is possible to obtain from a graph a competition graph by adding isolated vertices. The least number of isolated vertices needed for this procedure is called the competition number. Similarily every graph G can be embedded in an m-step graph $N_m(G')$ as an induced subgraph. The least number of vertices for such a graph G' is called the embedding number.

The embedding number was investigated by Boland, Brigham and Dutton in [2] and [3], based on the introduction of open neighborhood graphs by Acharya and Vartak [1].

Similarly to m-step graphs there is a generalization for competition graphs called m-step competition graph introduced by Cho, Kim and Nam [5] in 2000. The m-step competition graph of a digraph D is defined on the same vertex set and has edges xy if x and y have a common m-step pray, that is a vertex v with directed paths of length m from x to v and from y to v. They also introduced the m-step competition number. Further work on this definition was done by Helleloid [10] in 2004 investigating connected triangle-free m-step competition graphs, by Ho [11] in 2005 introducing same-step and any-step competition graphs and by Zhao and Chang in 2009 examining the m-step competition number of paths and cycles.

Determining the competition number appears to be a difficult problem: In 1971 Stephen A. Cook [7] published his paper on the concept of NP-completeness. Based on this Richard M. Karp [12] took 21 well-known problems - for which there were (and still are) no deterministic polynomial algorithms found - and proved their NP-completeness. Using these results James Orlin [16] was able to prove the NP-completeness of determining minimal edge-clique-covers (ECCs) in 1977 by reducing this problem amongst others to Karps chromatic number problem. Robert J. Opsut [15] then showed 1982 that the ECC problem is reducible to computing the competition number of graphs. That means, if there was a deterministic polynomial algorithm for computing the competition number, then the infamous equation P = NP would be solved.

2.3 Overview

In the following I want to give some detailed examples in Section 3, namely the descriptions of m-step graphs of well-known graph classes; paths, cycles, wheels and bipartite graphs. Then I will discuss some basic graph properties in Section 4 or to be more specific, I will answer some questions on how much these graph properties are preserved by the m-step function. As a first step in this, injectivity and surjectivity of the m-step function will be discussed. After that the minimum degree is a perfect example on how a graph property can be preserved by the m-step function. Two more of such interesting properties are connectivity and hamiltonicity, which got their own chapters 5 and 6. Finally I will have some conclusions, summaries and open problems in Section 7.

3 Particular Graph Classes

In this chapter I will describe particular m-step graphs, namely the m-step graphs of paths, cycles, wheels and bipartite graphs. Examining these graph classes will give us a basic idea on how to work with m-step graphs, so that we can rely on these results in the further chapters. Considering the complete graph K_n with n vertices, for example, its m-step graph is still K_n, respecting the condition $2 \leq m < n$ given in the introduction. Therefore by Proposition 1.3 we can conclude, for example, that any supergraph of K_n has again K_n in its m-step graph.

3.1 Paths

P_n is a path of length $n-1$ with n vertices.

Proposition 3.1. *Let $d \in [0, m-1]$ with $d \equiv n \mod m$. The m-step graph $N_m(P_n)$ consists of m paths; d of those paths have $\left\lceil \frac{n}{m} \right\rceil$ vertices, the other paths have $\left\lfloor \frac{n}{m} \right\rfloor$ vertices, i.e.*

$$N_m(P_n) = d \cdot P_{\lceil \frac{n}{m} \rceil} + (m-d) \cdot P_{\lfloor \frac{n}{m} \rfloor},$$

or by substitution $n = m \cdot k + d$ for any $k \in \mathbb{N}$ and $d \in [0, m-1]$ this is

$$N_m(P_{m \cdot k+d}) = d \cdot P_{k+1} + (m-d) \cdot P_k.$$

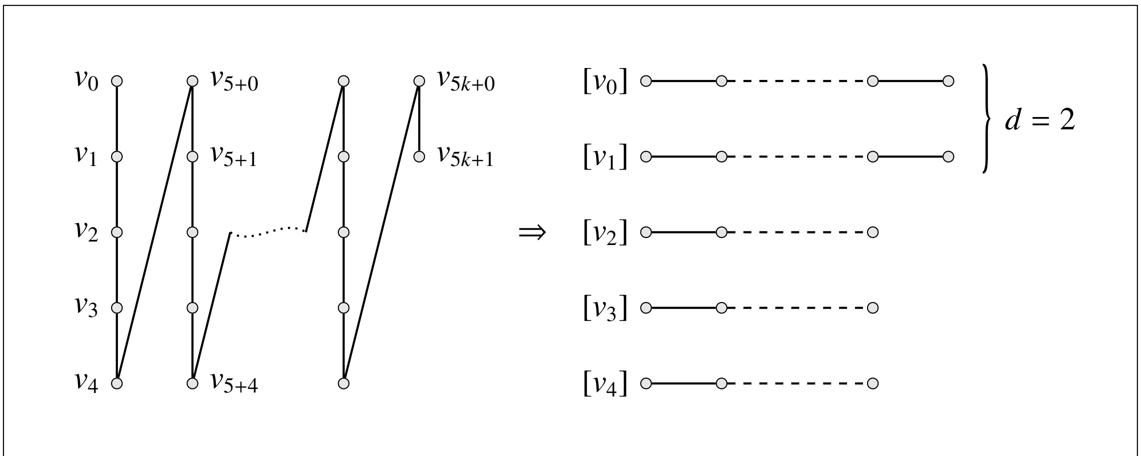

Figure 2: **The 5-step graph $N_5(P_{5k+2})$ consists of five paths.**

Proof. Let $n = m \cdot k + d$ and $P_n = v_0 \ldots v_{m-1} \ldots v_{n-1}$. Since there are at least m vertices there is a partition of V containing m subsets $\{[v_0], \ldots, [v_{m-1}]\}$ with

$$[v_i] := \{v_k \in V \mid k \equiv i \mod m\}.$$

The induced subgraphs in $N_m(P_n)$ having vertices $[v_i]$ are paths.

- There are two possibilities for the number of vertices, which is caused by

$$[v_i] = \begin{cases} v_i, v_{i+m}, v_{i+2m}, \ldots, v_{i+km} & \text{if } 0 \le i < d \\ v_i, v_{i+m}, v_{i+2m}, \ldots, v_{i+(k-1)m} & \text{if } d \le i < m \end{cases}.$$

  Therefore we obtain $|[v_i]| = \left\lfloor \frac{n-i}{m} \right\rfloor$, that is

$$|[v_i]| = \begin{cases} k + 1 = \left\lfloor \frac{n}{m} \right\rfloor + 1 & \text{if } 0 \le i < d \\ k = \left\lfloor \frac{n}{m} \right\rfloor & \text{if } d \le i < m \end{cases}.$$

  However, since $d = 0$ always follows the second case, we can rewrite the first case by using $\left\lceil \frac{n}{m} \right\rceil$ instead of $\left\lfloor \frac{n}{m} \right\rfloor + 1$.

- The edges induced are along the path $v_i v_{i+m} v_{i+2m} \ldots, v_{i+km}$.

- In addition these paths are not interconnected, because there can be no path of length m from v_i to v_j in $P_{m \cdot k+d}$ with $i \not\equiv j \mod m$.

Altogether we obtain d paths each with $\left\lceil \frac{n}{m} \right\rceil$ vertices and $m - d$ paths each with $\left\lfloor \frac{n}{m} \right\rfloor$ vertices.

□

3.2 Cycles

C_n is a cycle with n vertices, say $V(C_n) = \{v_0, \ldots, v_{n-1}\}$, $v_i v_{i+1} \in E(C_n)$ and indices are taken modulo n. For convenience C_1 denotes a single vertex and C_2 denotes two connected vertices instead of a real cycle.

Proposition 3.2. *Let $g = \gcd(m, n)$. The m-step graph of C_n consists of g cycles of equal length,*

$$N_m(C_n) = g \cdot C_{\frac{n}{g}}.$$

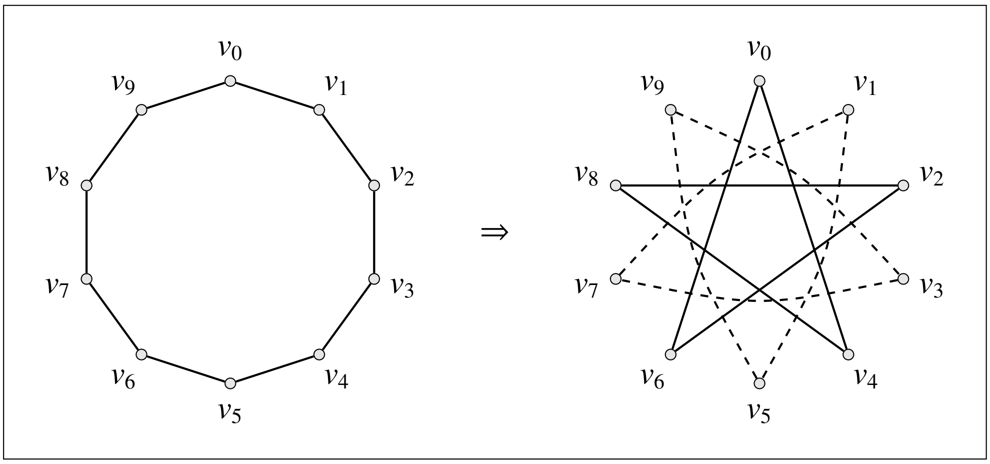

Figure 3: **The 4-step graph $N_4(C_{10})$ consists of two cycles C_5.**

Proof. There is a partition of $V(C_n)$ containing g subsets $\{[v_0], \ldots, [v_g]\}$ with

$$[v_i] := \{v_k \mid k \equiv i \mod g\} \subseteq V.$$

The induced subgraphs in $N_m(C_n)$ having vertices $[v_i]$ are cycles.

- The number of vertices $|[v_i]|$ is the least $k \in \mathbb{N}$ such that

  $$i + k \cdot m \equiv i \mod n.$$

  By subtracting i on both sides and dividing by g we obtain

  $$k \cdot \frac{m}{g} \equiv 0 \mod \frac{n}{g}.$$

  Because $\frac{m}{g}$ and $\frac{n}{g}$ are coprime, the least such k is exactly $\frac{n}{g}$.

- The edges induced are along the cycle $v_i v_{i+m} v_{i+2m} \ldots v_{i+\frac{n}{g}m}$ with $v_{i+\frac{n}{g} \cdot m} = v_i$ (indices taken modulo n).

- Let $[v_i]$ and $[v_j]$ be any two distinct sets of vertices. The cycles are not interconnected. This is proven by contradiction. If there was an edge $\{v_{i+a \cdot g}, v_{j+b \cdot g}\} \in N_m(C_n)$ $(a, b \in \mathbb{N})$ we would obtain $i + a \cdot g - (j + b \cdot g) \equiv 0 \mod m$ which means $i - j \equiv 0 \mod g$ and thus $[v_i] = [v_j]$.

Altogether we obtain g cycles each with $|[v_i]| = \frac{n}{g}$ vertices in $N_m(C_n)$.

□

3.3 Wheels

A wheel W_n is a graph with one center vertex connected to each vertex of a cycle of n vertices. Because of this notation $n = |V| - 1$ and $m \leq n$.

Proposition 3.3. *The m-step graph of a wheel W_n is the complete graph K_{n+1},*

$$N_m(W_n) = K_n.$$

Proof. Let $V(W_n) = \{v_0, v_1, \ldots, v_n\}$ with center vertex v_0 and circle $v_1, \ldots, v_n$. It is sufficient to show, that v_1 has paths of length m to each other vertex. Consider the following three cases showing $v_1 v_i \in E(N_m(W_n))$ for $i = 0$, $2 \leq i < m$ or $m \leq i \leq n$.

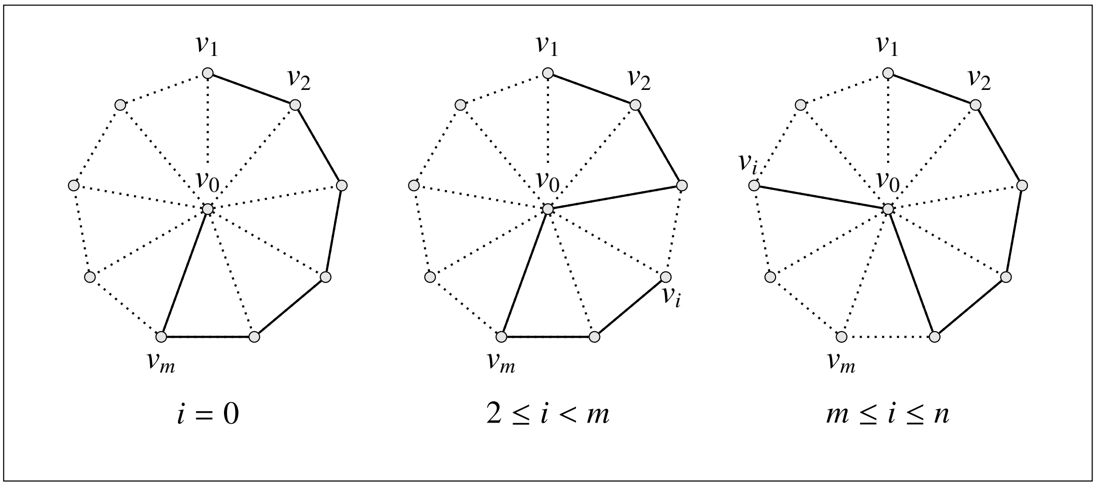

Figure 4: **Case distinction of the proof for wheels W_n, finding $v_1 v_i$-paths.**

- Let $i = 0$. Then $v_1 \ldots v_m v_0$ is a path of length m in W_n.
- Let $2 \leq i < m$. Then $v_1 \ldots v_{i-1} v_0 v_m v_{m-1} \ldots v_i$ is a path of length m in W_n.
- Let $m \leq i \leq n$. Then $v_1 \ldots v_{m-1} v_0 v_i$ is a path of length m in W_n.

Therefore $p_m(v_1) = V \setminus \{v_1\}$. Because of the symmetry in a wheel, it follows $p_m(v_i) = V \setminus \{v_i\}$ for $1 \leq i \leq n$. And by the symmetry of p_m, from $v_0 \in p_m(v_i)$ for $v_i \in V \setminus \{v_0\}$ it follows $p_m(v_0) = V \setminus \{v_0\}$.

□

3.4 Complete Bipartite Graphs

For a bipartite graph $G = (A \cup B, E)$ with $A \cap B = \emptyset$ let $a = |A|$, $b = |B|$. Without loss of generalization assume $a \leq b$. The complete bipartite graph $K_{a,b}$ is a bipartite graph with all possible edges $E = \left\{a_i b_j \mid a_i \in A \wedge b_j \in B\right\}$. The graph without edges having n vertices is denoted by I_n.

Proposition 3.4. *The m-step graph of the complete bipartite graph is*

$$N_m(K_{a,b}) = \begin{cases} K_{a,b} & \text{if } m \text{ is odd and } m < 2a \\ K_a + K_b & \text{if } m \text{ is even and } m < 2a \\ I_a + K_b & \text{if } m = 2a \text{ and } a < b \\ I_{a+b} & \text{otherwise} \end{cases}.$$

Proof. Since any path of length m in $K_{a,b}$ is alternating on A and B it can be written in exactly one of the following three notations:

$$\begin{aligned} P_{AA} &= a_0 b_1 a_2 b_3 \dots b_{m-1} a_m && \text{if } m \text{ is even} \\ P_{AB} &= a_0 b_1 a_2 b_3 \dots a_{m-1} b_m && \text{if } m \text{ is odd} \\ P_{BB} &= b_0 a_1 b_2 a_3 \dots a_{m-1} b_m && \text{if } m \text{ is even} \end{aligned}$$

	A	B
P_{AA}	$\frac{m}{2} + 1$	$\frac{m}{2}$
P_{AB}	$\frac{m+1}{2}$	$\frac{m+1}{2}$
P_{BB}	$\frac{m}{2}$	$\frac{m}{2} + 1$

Table 1: **Number of vertices of A and B traversed by paths P_{AA}, P_{AB} and P_{BB}.**

Table 1 describes the number of vertices of A and B traversed by each path P_{AA}, P_{AB} and P_{BB}. Let $a_0, a_m \in A$ and $b_0, b_m \in B$ be arbitrarily chosen vertices. The following existence propositions are true, because $b \geq a$ and $K_{a,b}$ is complete.

- P_{AA} exists if and only if $a \geq \frac{m}{2} + 1$, i.e. $m \leq 2a - 2$ and m is even.
- P_{AB} exists if and only if $a \geq \frac{m+1}{2}$, i.e. $m \leq 2a - 1$ and m is odd.
- P_{BB} exists if and only if $a \geq \frac{m}{2}$ and $b > a$, i.e. $m \leq 2a$ and $b > a$ and m is even.

For even m the m-step graph $N_m(K_{a,b})$ is induced by paths of type P_{AA} and P_{BB}. For odd m the m-step graph $N_m(K_{a,b})$ is induced only by paths of type P_{AB}. Therefore for even m we obtain the union of K_a and K_b; for odd m we obtain again the bipartite graph $K_{a,b}$. Only for $m = 2a$ and $b > a$ the paths P_{BB} do exist while paths P_{AA} do not exist; thus we obtain in this case the union of I_a and K_b.

□

A star is a graph with one center vertex and n additional vertices connected to its center, thus a star is $K_{1,n}$ and

$$N_m(K_{1,n}) = \begin{cases} K_{1,n} & \text{if } m = 1 \text{ (trivial)}, \\ K_n + I_1 & \text{if } m = 2, \\ I_{1+n} & \text{otherwise.} \end{cases}$$

4 Basic Results for Arbitrary Graphs

In this chapter I will present some basic results on the structure of m-step graphs. After investigating injectivity and surjectivity of the m-step function I will answer some other questions similar to that of surjectivity. Then we will discuss a lower bound for the minimum degree of an m-step graph. After that I will finish this section by investigating some isomorphism problems, that ask for characterizations of graphs G such that the equations $N_m(G) = K_n$, $N_m(G) = G$ or $N_m(G) = \overline{G}$ are fulfilled.

There are two elemental questions concerning m-step graphs:

- If two graphs G_1 and G_2 have the same m-step graph $N_m(G_1) = N_m(G_2)$, does that imply $G_1 = G_2$?
- Is any graph $G \in \mathcal{G}(V)$ an m-step graph? That is, for any graph G is there another $G' \in \mathcal{G}(V)$ such that $N_m(G') = G$?

We can define the m-step function as a function mapping from simple to simple graphs, i.e.

$$N_m : \mathcal{G}(V) \to \mathcal{G}(V), \qquad N_m : (V, E) \mapsto (V, \{xy \mid y \in p_m(x)\}).$$

With that the above questions ask for injectivity and surjectivity of this m-step function.

Proposition 4.1. *Let V be any (finite) vertex set and $m \in \mathbb{N}$ with $2 \leq m < |V|$. Then the function $N_m : \mathcal{G}(V) \to \mathcal{G}(V)$ is neither injective nor surjective.*

Proof. Since $N_m(K_2) = N_m(I_2) = I_2$ the function N_m is not injective. Furthermore the domain and codomain are equal and finite, therefore the range of N_m has less elements than its codomain and thus N_m is not surjective.

□

The restriction to finite graphs makes this proof easier, however, as we will realize later, there are graphs that are not m-step graphs no matter how many vertices (or edges) a graph is allowed to have.

Concerning the trivial cases of m-step graphs as described in the introduction, we can complete the above proposition by the following.

- If $m = 1$, then $N_m(G) = G$; thus N_m is bijective.

- Otherwise if $m \geq |V|$, then $N_m(G)$ is empty. Therefore N_m is bijective for $|V| \leq 1$ and neither injective nor surjective for $|V| \geq 2$.

In order to develop a better understanding we will weaken the condition of surjectivity and examine the results. Let G be an arbitrary graph in $\mathcal{G}(V)$. Surjectivity asks for a graph $G' \in \mathcal{G}(V)$ such that $N_m(G') = G$. By adding t vertices to G' we have $G' \in \mathcal{G}(V \cup \{v_1, \ldots, v_t\})$. However, that makes $N_m(G') \neq G$ in any case, because they are defined on different vertex sets. That is why we should ask for the following questions.

1. Is there a graph G' such that $N_m(G')$ contains only G and t isolated vertices?
2. Is there a graph G' such that $N_m(G')$ contains G as a component?
3. Is there a graph G' such that $N_m(G')$ contains G as an induced subgraph?

The second and thus the third question as well will be answered positivly by the following proposition.

Proposition 4.2. *For any graph $G \in \mathcal{G}(V)$ there is a $t \in \mathbb{N}_0$ such that there exists a graph $G' \in \mathcal{G}(V \cup \{v_1, \ldots, v_t\})$ with $N_m(G')$ containing G as a component.*

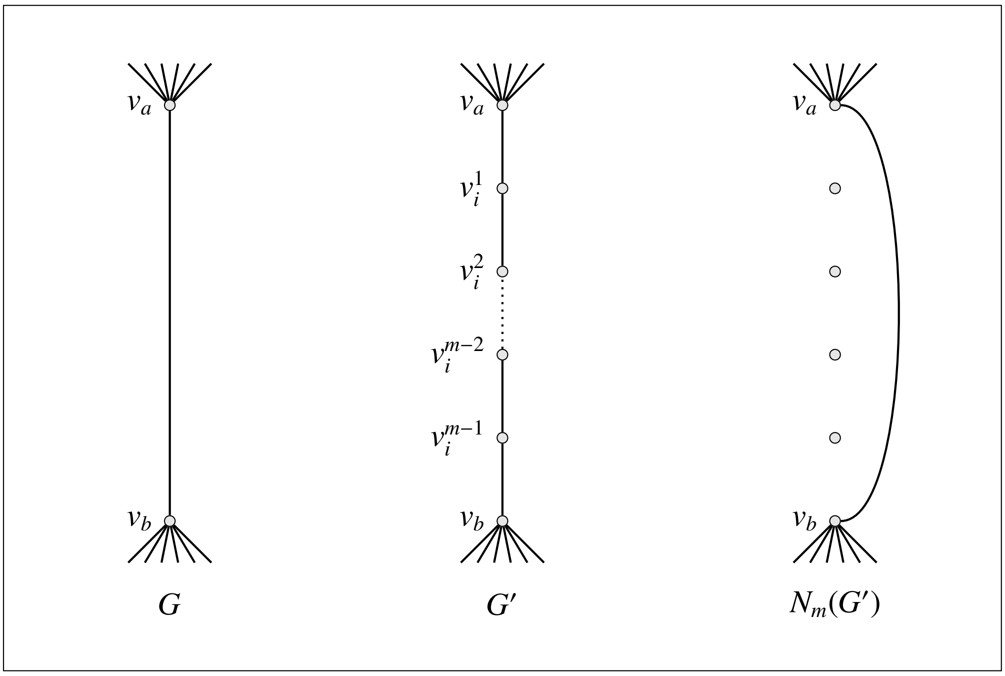

Figure 5: **Subdividing edges such that G is a component of $N_m(G')$ ($e_i = v_a v_b, a < b$).**

Proof. Let $G = (V, E)$ be a simple graph with vertices $V = \{v_1, \dots, v_n\}$ and edges $E = \{e_1, \dots, e_k\}$. Then $G' = (V', E')$ is constructed by subdividing each edge in G into m edges.

$$
\begin{aligned}
V' &= V \cup \left\{v_i^j \mid i = 1, \dots, k, j = 1, \dots, m-1\right\} \\
E' &= \left\{v_i^j v_i^{j+1} \mid i = 1, \dots, k, j = 1, \dots, m-2\right\} \\
&\cup \left\{v_a v_i^1, v_i^{m-1} v_b \mid v_a v_b = e_i, a < b, i = 1, \dots, k\right\}
\end{aligned}
$$

Now $N_m(G')$ contains G as an induced subgraph: By construction the vertex set V is a subset of V'. Let $e_i = v_a v_b \in E$ with $a < b$ be an arbitrary edge in G. Since G' contains a path of length m from v_a to v_b, namely $v_a v_i^1 v_i^2 \dots v_i^{m-1} v_b$, this edge e_i is in $E(N_m(G'))$ as well. In addition, any path of length m in G' with end vertices in $V(G)$ by construction has a corresponding edge in $E(G)$. Therefore G is an induced subgraph of $N_m(G')$.

It is left to prove, that G is a component of $N_m(G)$, which means, that the subgraph is not connected to any vertex in $V' \setminus V = \left\{v_i^j \mid i = 1, \dots, k, j = 1, \dots, m-1\right\}$. However, by construction any path of length m starting in $v_a \in V(G)$ ends in $v_b \in V(G)$. Thus because of symmetry, there can be no path from v_a to $v_i^j \in V(G)$ of length m, and there is no edge $v_a v_i^j \in N_m(G')$.

□

Furthermore, if G is a component of $N_m(G')$ then it is also an induced subgraph, thus positivly answering the third question. In that case the least number $|V(G')| = |V(G)| + t$ is called the embedding number of G. Determining the embedding number seems not to be an easy problem. Boland, Brigham and Dutton have done research on this for neighborhood graphs N_2 in [2] and [3]. However, the NP-completeness has not been proven yet, and the embedding number of m-step graphs $N_m(G), m \geq 3$ was not investigated yet.

The first question however has a negative answer. For example let $m = 2$, then $G = P_2 = acb$ plus any number of isolated vertices is not a neighborhood graph. For those two edges ac and bc there must be two vertices v_1, v_2 adjacent to the ends of these edges. If they are identical $v_1 = v_2$ then there is an edge $ab \in E(N_2(G))$ induced by the path av_1b. This yields a triangle instead of a path. If the vertices are not identical $v_1 \neq v_2$, then they have a common neighbor c and thus are connected in the neighborhood graph. For arbitrary $m \geq 3$ the graph G' with $N_m(G') = P_2 + I_t$ can be constructed as shown in Figure 6.

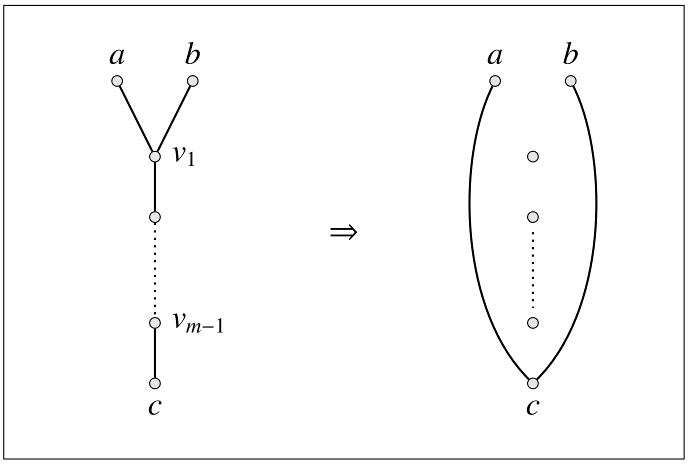

Figure 6: **An m-step graph consisting of $P_2 = acb$ and isolated vertices for $m \geq 3$.**

However, if $G = P_3$ there is no graph G' with $N_m(G') = P_3 + I_t$.

Proposition 4.3. *Let $m \geq 2$ and $t \in \mathbb{N}_0$. Then there is no graph G' such that $N_m(G') = P_3 + I_t$.*

Proof. Let $P_3 = acbd$ a path of length three. The subpath acb must be a fork in G' as can be seen in Figure **??**: Let ac be induced by the path $a = a_0a_1 \ldots a_m = c$ and bc induced by the path $b = b_0b_1 \ldots b_m = c$. It follows, that there must be a minimal $i < \frac{m}{2}$ with $a_i = b_i$; otherwise this would yield edges in $N_m(G')$ that are not in $acbd$. Now there must be a path of length m in G' inducing the edge bd. However, by a case distinction on where this path must diverge from the other paths, in any case there is a fourth edge or a triangle induced. Therefore we obtain contradictions such that there is no G' with $N_m(G') = P_3 + I_t$.

□

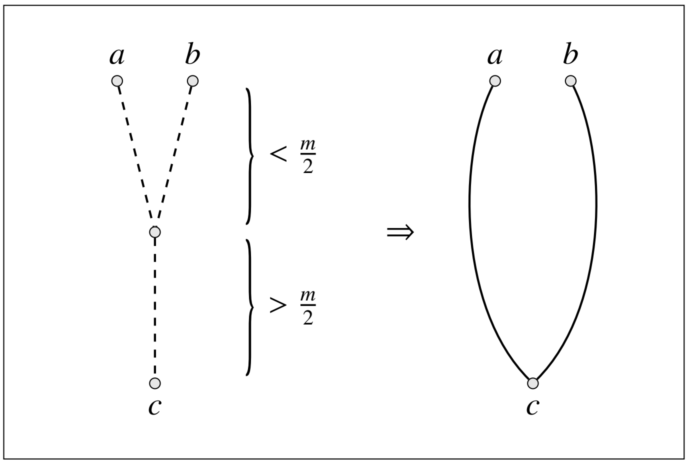

Figure 7: **Any bd-path in G implies edges in $N_m(G')$, that are not in $acbd$.**

4.1 Minimum Degree

A lower bound for the minimum degree of m-step graphs is given by

Theorem 4.4. *Let $G = (V, E)$ be a graph with minimum degree $\delta(G)$ and $m \in \mathbb{N}$ with $2 \leq m \leq \delta(G)$. Then we obtain for the m-step graph $N_m(G)$*

$$\delta(N_m(G)) \geq \delta(G) - 1. \quad (1)$$

Proof. Without loss of generality assume that G is connected (otherwise the following considerations can be made seperatly for each component). We have $|V| > \delta(G)$ and because of $m \leq \delta(G)$ there is a path $P = v_1 \dots v_m$ of length $m - 1$ in G for an arbitrarily chosen start vertex $v_1 \in V$. In the following we show the existence of $\delta(G) - 1$ paths of length m from v_1 to pairwise distinct end vertices, which proves (1).

(a) Let $j = |\{v_i v_m \in E \mid i \in \{1, \dots, m-2\}\}|$, that is the number of edges from any vertex v_i in the path (except from v_{m-1}) to the end vertex v_m. Then v_m has at least $\delta(G) - (j + 1)$ neighbors $a_1, \dots, a_{\delta(G)-j-1} \notin V(P)$. Now the path Pa_i from v_1 to a_i is of length m in G, thus $v_1 a_i \in E(N_m(G))$ for $i = 1, \dots, \delta(G) - j - 1$ (see Figure (8)).

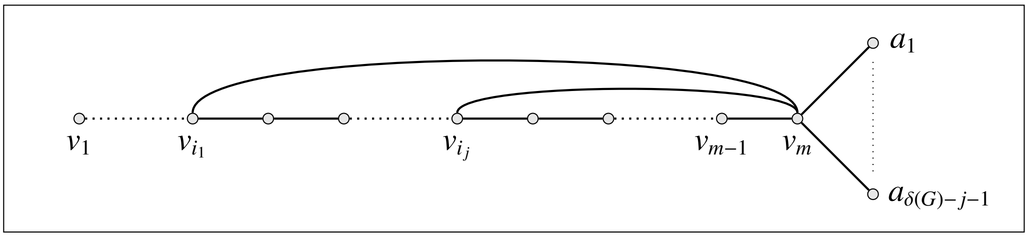

Figure 8: **Path $v_1 \dots v_m$, neighbors of v_m and $j = 2$ edges from v_m back to the path.**

(b) Now consider the j vertices $v_{i_1}, \dots, v_{i_j} \in V(P)$ with $i_t \in \{2, \dots, m-1\}$, $v_{i_t-1} v_m \in E$ and $t = 1, \dots j$. Because of $\delta(G) \geq m$ there are (not necessarily distinct) vertices $w_t \notin V(P)$ with $w_t v_{i_t+1} \in E$ and $t = 1, \dots, j$. For each $t \in \{1, \dots, j\}$ we distinguish three cases (see Figure (9)).

(i) Let $b_t := v_{i_t+2}$ and $w_t v_m \in E$ (that is $w_t = a_i, i \in \{1, \dots, \delta(G) - j - 1\}$). Then the path $P_1 = v_1 \dots v_{i_t} v_{i_t+1} w_t v_m \dots b_t$ has length m, and thus $v_1 b_t \in E(N_m(G))$.

(ii) Let $b_t := w_t$ and $w_t v_m \notin E$. If there is no other vertex v_{i_r} with $i_r > i_t$ and $\{v_{i_r+1}, w_t\} \in E$ then the path $P_2 = v_1 \dots v_{i_t} v_m \dots v_{i_t+1} b_t$ has length m and $\{v_1, b_t\} \in E(N_m(G))$. Note that this case appears at most once for each vertex w_t.

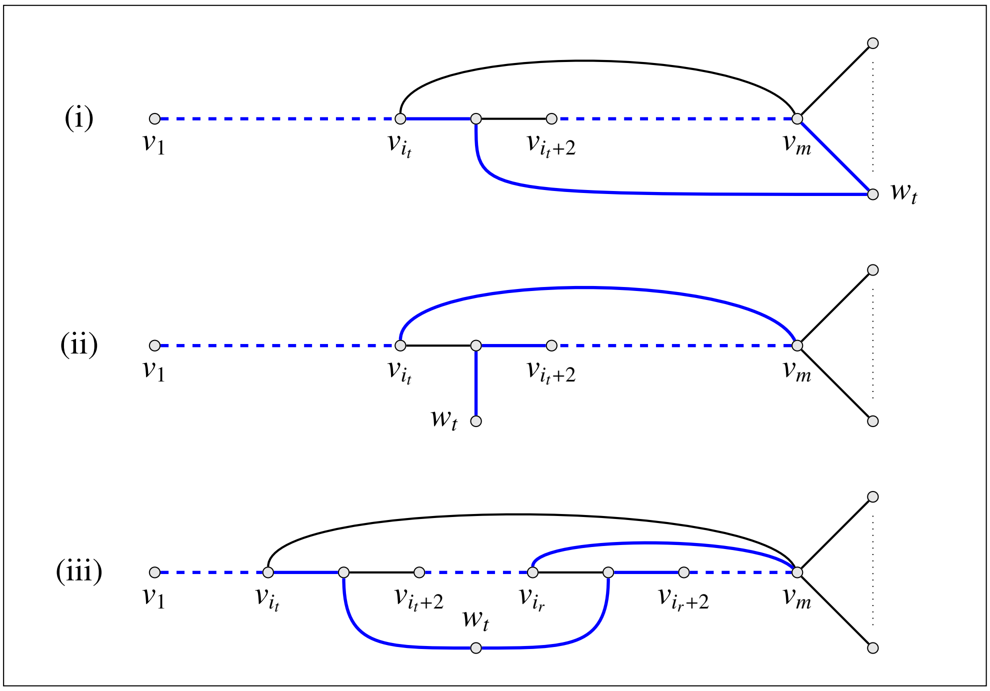

Figure 9: **Case distinction of the proof for the minimum degree.**

(iii) Otherwise there is a vertex v_{i_r} with $i_r > i_t$ and $\{v_{i_r+1}, w_t\} \in E$. Let $b_t := v_{i_t+2}$. Then the path $P_3 = v_1 \ldots v_{i_t} v_{i_t+1} w_t v_{i_r+1} \ldots v_m v_{i_r} \ldots b_t$ is of length m and $v_1 b_t \in E(N_m(G))$.

Summarizing the results we obtain the vertices $a_1, \ldots, a_{\delta(G)-j-1}$ and $b_1, \ldots, b_j$. In cases (i) and (iii) $b_t = v_{i_t+2}$ and in case (ii) $b_t = w_t$. Furthermore these vertices are pairwise distinct and we obtain $\delta(G) - 1$ edges $\{v_1, a_i\}, \{v_1, b_t\} \in E(N_m(G))$, this completes the proof.

□

This lower bound for the minimum degree is sharp.

- For K_n we have $\delta(K_n) = n - 1$ and $\delta(N_m(K_n)) = 0$ for any $m \geq n > \delta(K_n)$.
- There are graphs with $\delta(N_m(G)) = \delta(G) - 1$. Let $m \in \mathbb{N}$ and $m \geq 2$. Graph $G = (V, E)$ is the union of a tree and a complete bipartite graph; the tree has the root vertex v and all leafs are at depth $m - 1$, the inner nodes have degree of m and all the leaf vertices are connected to the vertices $\{y_1, \ldots, y_{m-1}\}$. See Figure 10 for an example. To be precise, the construction is formally given by

$$V = \bigcup_{i=0}^{m} L_i, \quad L_i = \begin{cases} \{v\} & \text{if } i = 0 \\ \left\{v_p \mid p \in [1, m] \times [1, m-1]^{i-1}\right\} & \text{if } 0 < i < m \\ \{y_k \mid k \in [1, m-1]\} & \text{if } i = m \end{cases}$$

and the set of edges E is defined by the following four conditions

(i) $\{vv_1, \ldots, vv_m\} \subseteq E$,

(ii) $\forall i \in [1, m-2]\ \forall w \in L_i\ : w = v_{p_1,\ldots,p_i} \rightarrow \left\{wv_{p_1,\ldots,p_i,1}, \ldots, wv_{p_1,\ldots,p_i,m-1}\right\} \subseteq E$,

(iii) $\forall w \in L_{m-1} : \{wy_1, \ldots, wy_{m-1}\} \subseteq E$,

(iv) there is no other edge in E than those given in (i), (ii) and (iii).

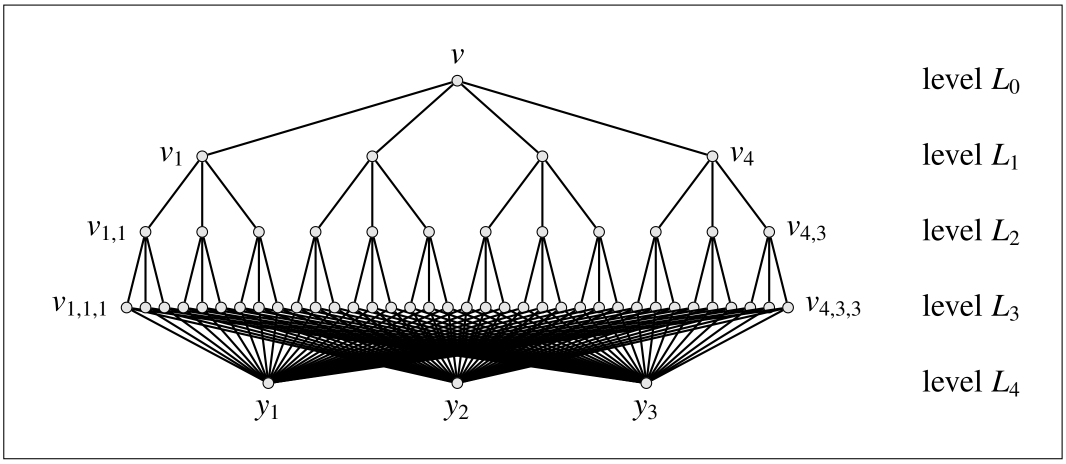

Figure 10: **Example for** $m = \delta(G) = 4$ **and** $\delta(N_m(G)) = \delta(G) - 1$**.**

The graph G has the minimum degree $\delta(G) = m$. Because of Theorem 4.4 the minimum degree of its m-step graph is $\delta(N_m(G)) \geq \delta(G) - 1$. Considering the start vertex $v \in G$ we notice that every path of length m is of the type $vv_{p_1}v_{p_1,p_2} \ldots v_{p_1,p_2,\ldots,p_{m-1}}y_k$. Therefore $N_m(v) = \{y_1, \ldots, y_{m-1}\}$ and thus $\delta(N_m(G)) = m - 1$.

4.2 Isomorphism Problems

Brigham and Dutton [4] gave characterizations for graphs G such that $N_2(G) \cong K_n$ and $N_2(G) \cong G$. Furthermore they described new results on the more difficult problem $N_2(G) \cong \overline{G}$. With respect to m-step graphs, these problems can be generalized by the following equations:

- $N_m(G) \cong K_n$,
- $N_m(G) \cong G$ or
- $N_m(G) \cong \overline{G}$.

Let us first consider the equation $N_m(G) \cong K_n$ (remember that $|V| = n > m \geq 2$).

Proposition 4.5. *Necessary conditions for $N_m(G) = K_n$ are:*

(i) G is connected,

(ii) diameter $d(G) \leq m$,

(iii) if $v \in V(G)$ is a cut vertex seperating A, B, then $|V(A)| > m$ and $|V(B)| > m$,

(iv) there is no bridge in G,

(v) each $e \in E(G)$ is part of a cycle of length $m + 1$.

Proof. If $d(G) > m$, then there are two vertices $x, y \in V(G)$, such that there is no xy-path of length m, thus $xy \notin E(N_m(G))$. Therefore (ii) is necessary, which implies, that (i) is necessary too. Condition (iii) is necessary, because any xy-path in G requires at least $m + 1$ vertices and can not visit a cut vertex twice. Any edge $e = xy \in E(G)$ requires an xy-path of length m in G, otherwise $e \notin E(N_m(G))$. Therefore e is part of a cycle of length $m + 1$, and (v) is necessary, which implies, that (iv) is necessary too.

□

Obviously these conditions are not sufficient for $m \geq 3$; C_{m+1} for example fulfills these necessary conditions, but $N_m(C_{m+1}) \cong C_{m+1} \ncong K_{m+1}$.

Let us now consider the equation $N_m(G) \cong G$. For $m = 2$ Brigham and Dutton have shown, that every component of G is a complete graph on other than two vertices or an odd cycle. However, for $m \geq 3$ we obtain only sufficient but not necessary conditions by generalizing these conditions. If $n > m$ then $N_m(K_n) = K_n$, and if also $\gcd(m, n) = 1$ then $N_m(C_n) \cong C_n$, but another example is given in Figure 11 showing a spiked cycle isomorphic to its 3-step graph. Similarily for any odd m there is a spiked cycle $G = C_{2m-2} + x + xv_0$, such that $N_m(G) \cong G$. Finding elegant conditions for problems with $m \geq 3$ remains an open problem.

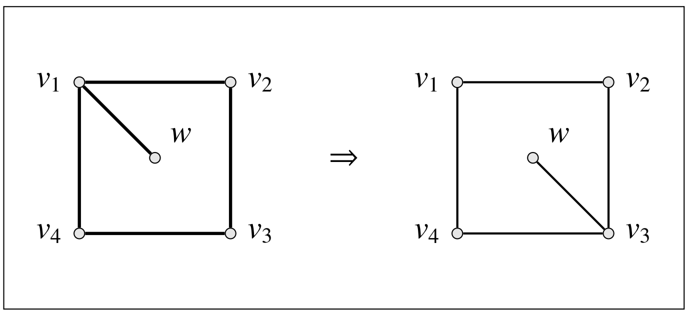

Figure 11: **The spiked cycle on five vertices is isomorphic to its 3-step graph.**

5 Connectivity

A necessary condition for the connectivity of an m-step graph $N_m(G)$ is the connectivity of G, as described in the following proposition.

Proposition 5.1. *If $N_m(G)$ is connected, then G is also connected.*

Proof. Let $x, y \in V$ be any arbitrary vertices. Since $N_m(G)$ is connected, there is a path P from x to y in $N_m(G)$. An edge of this path is induced by a path of length m in G. Thus there is a walk from x to y in G and G is connected.

□

Now the other way round is more difficult. Does connectivity of G imply also the connectivity of $N_m(G)$? From Section 3.1 we already know, that a path P is split up into m paths in $N_m(P)$. Furthermore $N_m(K_{1,n}) = I_{n+1}$ if $m \geq 3$ (see 3.4), thus there is no boundary for the number of components of an m-step graph $N_m(G)$ of a connected graph G. However, one might come to the idea, that if a graph is connected, large enough and contains certain subgraphs, then perhaps the connectivity of $N_m(G)$ is guaranteed. An example of such a subgraph is the cycle C_{m+1}. Since $N_m(C_{m+1}) = C_{m+1}$ remains connected this cycle induces connectivity of any supergraph. More general we obtain

Proposition 5.2. *Let G be connected and $H \subseteq G$ with $N_m(H)$ connected and having more than one vertex, then $N_m(G)$ is connected.*

Proof. Let $x, y \in V(G)$ be arbitrarily chosen. It is to prove, that there is an xy-path in $N_m(G)$.

(i) If $x, y \in V(H)$ then there is an xy-path in $N_m(G)$, because $N_m(H) \subseteq N_m(G)$ (by Proposition 1.3) and $N_m(H)$ is connected.

(ii) Let $x \notin V(H), y \in V(H)$. Since G is connected, there is a path P_1 from x to $v \in V(H)$ in G such that $\{v\} = V(P) \cap V(H)$ (see Figure 12). The length of P_1 is $k \cdot m + d$ with $k \in \mathbb{N}$ and $d \in [0, m-1]$. Now let $v_m \in V(H)$ such that there is a vv_m-path $vv_1 \ldots v_{m-1}v_m$ of length m in H. Then $xP_1v \ldots v_{m-d}$ is a path in G with a length divisible by m. Therefore there is a path from x to $v_{m-d} \in V(H)$ in $N_m(G)$. By extending this path with an $v_{m-d}y$-path as in (i) we obtain a walk from x to y in $N_m(G)$ and thus have an xy-path in $N_m(G)$.

(iii) Let $x, y \notin V(H)$ and $v \in V(H)$ arbitrarily chosen. By (ii) we obtain an xv-path and an yv-path, therefore there exists an xy-path in $N_m(G)$.

In any case there exists an xy-path in $N_m(G)$ for arbitrarily chosen $x, y \in V(G)$, therefore $N_m(G)$ is connected.

□

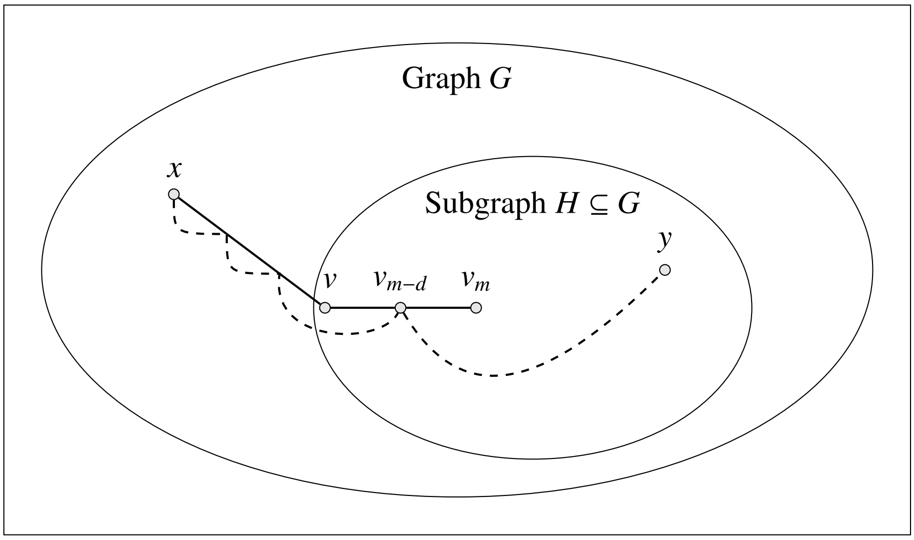

Figure 12: **Showing the existence of an xy-path for Proposition 5.2 (ii).**

A subgraph $H \subseteq G$ as in above proposition is called minimal, if there is no other subgraph $H' \subset H$ of at least two vertices such that $N_m(H')$ is connected. For $m = 2$ it is not difficult to see, that the only minimal graphs H with $N_2(H)$ connected are odd cycles:

- Odd cycles are connected and contain more than two vertices. Since N_2 preserves odd cycles (under isomorphism) Proposition **??** holds.
- Odd cycles are minimal. Proper connected subgraphs of odd cycles are paths, their m-step graphs are disconnected by Proposition 3.1.
- In order to show, that there are no minimal subgraphs $H \subseteq G$ other than odd cycles inducing connectivity in $N_2(G)$, assume G does not contain odd cycles. Then G is bipartite and by Proposition 3.4 $N_2(G)$ is disconnected.

By the same reason an odd cycle is required even though not sufficient for even $m \geq 4$. However, for odd $m \geq 3$ a minimal subgraph H with $N_m(H)$ does not require any cycles. To give an example I will show the connectivity of an acyclic graph, a caterpillar graph that is a tree having its leaf vertices within a distance of 1 from a central (longest) path.

Proposition 5.3. *For any odd $m \geq 3$ the m-step graph $N_m(G)$ of the following caterpillar graph $G = (V, E)$ (see Figure 13) is connected (and even contains a hamiltonian path):*

$$
\begin{aligned}
V &= \{a_i, b_i, c_i, d_i \mid i = 1, \ldots, m-1\}, \\
E &= \{a_i a_{i+1}, b_i b_{i+1}, c_i c_{i+1} \mid i = 1, \ldots, m-2\} \\
&\quad \cup \{b_i d_i \mid i = 1, \ldots, m-1\} \\
&\quad \cup \{a_{m-1} b_1, b_{m-1} c_1\}.
\end{aligned}
$$

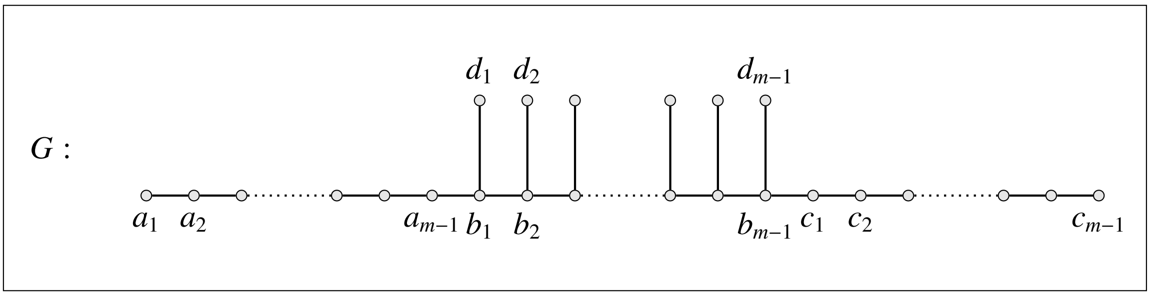

Figure 13: **The caterpillar graph of Proposition 5.3.**

Proof. For odd $m \geq 3$ the m-step graph $N_m(G)$ contains the following edges

$$
\begin{aligned}
E(N_m(G)) &= \{a_i b_{i+1}, b_i c_{i+1} \mid i = 1, \ldots, m-2\} \\
&\quad \cup \{a_i d_i, d_i c_i \mid i = 1, \ldots, m-1\} \cup \{c_1 a_{m-1}, d_1 d_{m-1}\}.
\end{aligned}
$$

Therefore $N_m(G)$ contains a hamiltonian path $P = b_1 P_1 a_{m-1} c_1 P_2 b_{m-1}$ (see Figure 14) with

$$
\begin{aligned}
P_1 &= (b_1 c_2 d_2 a_2)(b_3 c_4 d_4 a_4) \ldots (b_{m-2} c_{m-1} d_{m-1} a_{m-1}), \\
P_2 &= (c_1 d_1 a_1 b_2)(c_3 d_3 a_3 b_4) \ldots (c_{m-2} d_{m-2} a_{m-2} b_{m-1}).
\end{aligned}
$$

□

Note that the constructed path uses every edge in $E(N_m(G))$ except for $d_1 d_{m-1}$. Furthermore this caterpillar graph is minimal regarding the above definition, i.e. for any proper subgraph $H \subset G$ ($|(|V(H)) \geq 2$) the m-step graph $N_m(H)$ is disconnected.

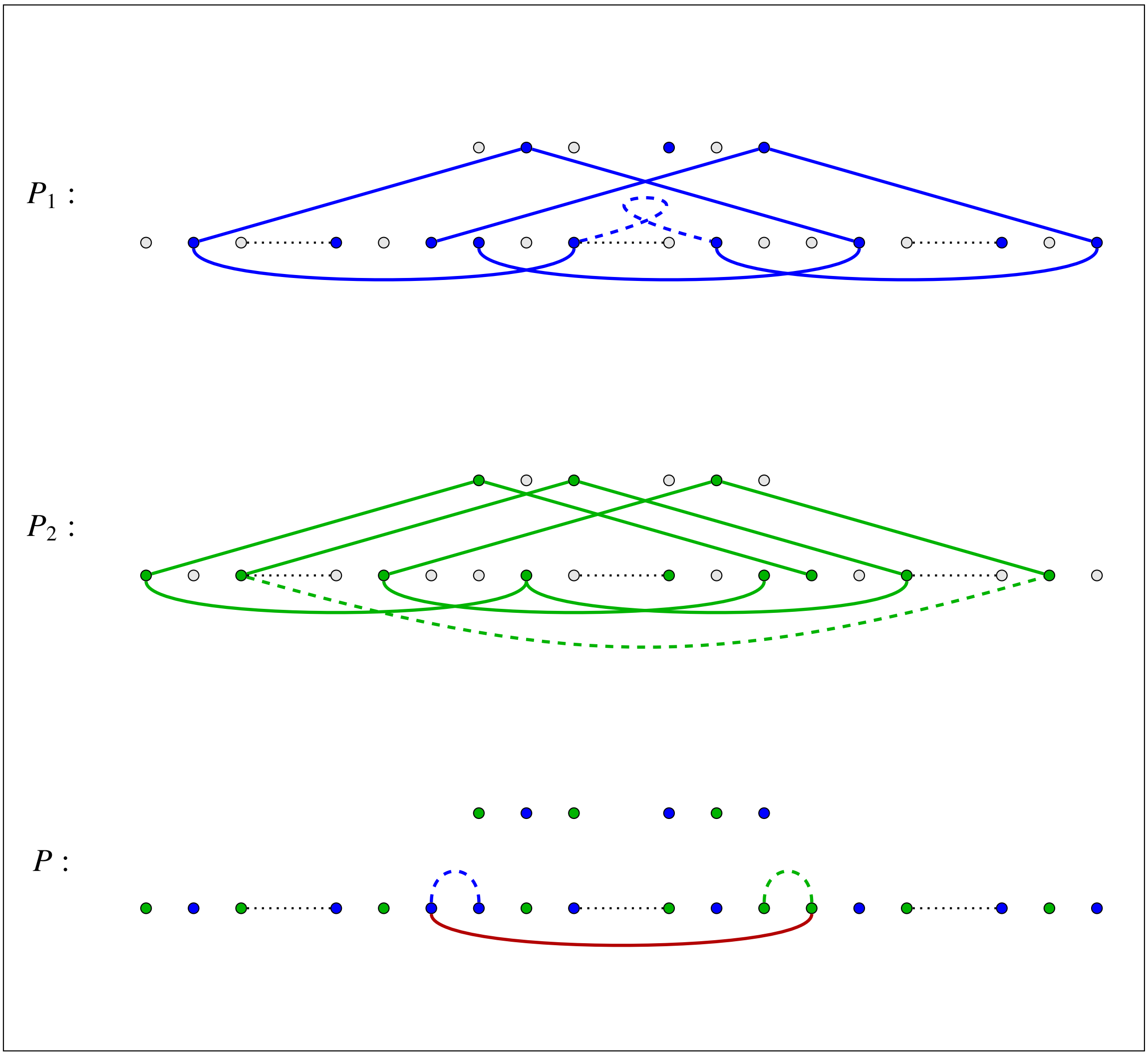

Figure 14: **Prooving the connectivity by finding a hamiltonian path in $N_m(G)$.**

6 Hamiltonicity

In this chapter I want to generalize some ideas on hamiltonicity concerning neighborhood graphs.

A simple result follows from theorem 4.4 and Dirac's theorem stating: Any simple graph G on $n \geq 3$ vertices is hamiltionian if each vertex has degree at least $\frac{n}{2}$.

Corollary 6.1. *Let $G = (V, E)$ be a simple graph, $n = |V| \geq 3$ and $m < n$. If $\delta(G) \geq \frac{n}{2} + 1$ then $N_m(G)$ is hamiltonian.*

Proof. From $\delta(G) \geq \frac{n}{2} + 1$ theorem 4.4 follows $\delta(N_m(G)) \geq \frac{n}{2}$, which implies a hamiltonian cycle according to Dirac.

□

In their paper Schiermeyer, Sonntag and Teichert [18] have proven some interesting propositions, answering the question on how $N_2(G)$ does inherit hamiltonicity properties from G. Their basic results are

Proposition 6.2. *Let $G = (V, E)$ be a graph and $N_2(G)$ its neighborhood graph.*

(i) If $|V|$ is odd and G is hamiltonian then $N_2(G)$ is hamiltonian.

(ii) If G is nonbipartite and hamiltonian then $N_2(G)$ contains a hamiltonian path.

(iii) If G has an odd spanning spiked cycle then $N_2(G)$ is hamiltonian.

(iv) If G is 1-hamiltonian then $N_2(G)$ is hamiltonian.

The first proposition can easily be generalized and proven; if $\gcd(m, |V|) = 1$ and G is hamiltonian then $N_m(G)$ is hamiltonian, because by Proposition 3.2 a hamiltonian cycle is congruent to a hamiltonian cycle in $N_m(G)$.

With that in mind one might come to the conclusion, that the odd spanning spiked cycle of 6.2 (iii) can be generalized to a spanning spiked cycle of length n with $\gcd(m, n) = 1$. However, this is not the case. A counterexample is the 3-step graph of the spiked cycle as already seen in Figure 11. Instead there are variants of spikes for which the m-step graph is hamiltonian:

- A non-cycle vertex connected to two cycle vertices at a given distance (Proposition 6.3).

- Two spikes at a given distance on the cycle (Proposition 6.4).
- Cycles of length m can be appended to cycle vertices (Proposition 6.5).
- Pairs of paths of length $m-1$ can be appended to one cycle vertex (Proposition 6.6).

Proposition 6.3. *Let $G = (V, E)$ with $V = \{v_0, \dots, v_{n-1}, w\}$. Furthermore let $E = \{wv_a, wv_b\} \cup \{v_i v_{i+1} \mid i = 0, \dots, n-1\}$ (indices are taken modulo n) such that $b - a \equiv m - 2 \mod n$ (or $a - b \equiv m - 2 \mod n$), then $N_m(G)$ is hamiltonian.*

Proof. Without loss of generalization assume $a = 1$ and $b = m - 1$ (otherwise this can be achieved by switching a and b or by rotation ($v_i \to v_{i+\tau}$, indices taken modulo n).

From $\gcd(m, n) = 1$ and Proposition 3.2 we obtain a cycle in $N_m(G)$ covering the cycle vertices $v_0, \dots, v_{n-1}$. To integrate the vertex w in this cycle there are three edges of importance (see Figure 15):

- $wv_0 \in E(N_m(G))$ since there is a path of length m in G, namely $wv_{m-1}v_{m-2}\dots v_1 v_0$.
- $wv_m \in E(N_m(G))$ since there is a path of length m in G, namely $wv_1 v_2 \dots v_{m-1} v_m$.
- $v_0 v_m \in E(N_m(G))$ is obtained from the induced cycle of C_N.

Now $(V, \{v_i v_{i+m} \mid i = 1, \dots, n-1\} \cup \{wv_0, wv_m\}) \subseteq N_m(G)$ is a hamiltonian cycle.

□

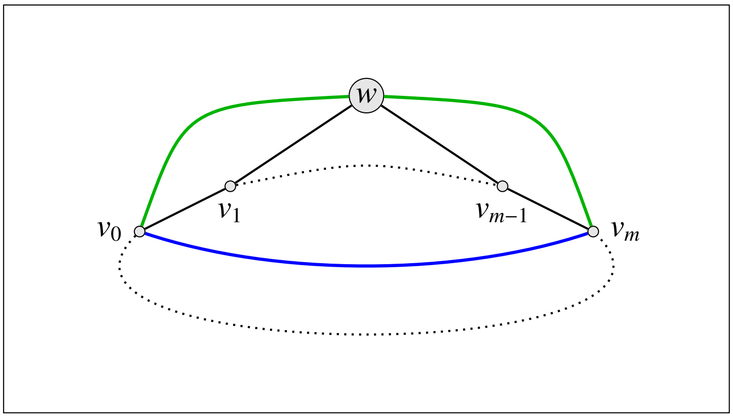

Figure 15: **Replacing $v_0 v_m$ (blue) by $v_0 w v_m$ (green) yields a hamiltonian cycle.**

In case of $m = 2$ the precondition $b - a \equiv m - 2 \mod n$ means $b = a$ and therefore $v_a = v_b$, yielding the familiar spike in Proposition 6.2 (iii).

Another possibility exists by using two spikes at a given distance:

Proposition 6.4. *Let* $G = (V, E)$ *with* $V = \{v_0, \ldots, v_{n-1}, x, y\}$. *Furthermore let* $E = \{xv_a, yv_b, xy\} \cup \{v_i v_{i+1} \mid i = 0, \ldots, n-1\}$ *(indices are taken modulo n) such that* $b - a \equiv m - 2 \mod n$ *(or* $a - b \equiv m - 2 \mod n$*), then* $N_m(G)$ *is hamiltonian.*

Proof. Similar to the proof of the preceding proposition, assume $a = 1$ and $b = m-1$ without loss of generelization. There are four edges of importance to construct a hamiltonian cycle:

- $v_0 y \in E(N_m(G))$, since $v_0 v_1 \ldots v_{m-2} v_{m-1} y$ is a path of length m in G.
- $v_m x \in E(N_m(G))$, since $x v_1 v_2 \ldots v_{m-1} v_m$ is a path of length m in G.
- xy, since $x v_1 v_2 \ldots v_{m-2} v_{m-1} y$ is a path of length m in G.
- $v_0 v_m \in E(N_m(G))$ is obtained from the induced cycle of C_N.

Therefore $(V, \{v_i v_{i+m} \mid i = 1, \ldots, n-1\} \cup \{xv_0, xy, yv_m\}) \subseteq N_m(G)$ is a hamiltonian cycle.

□

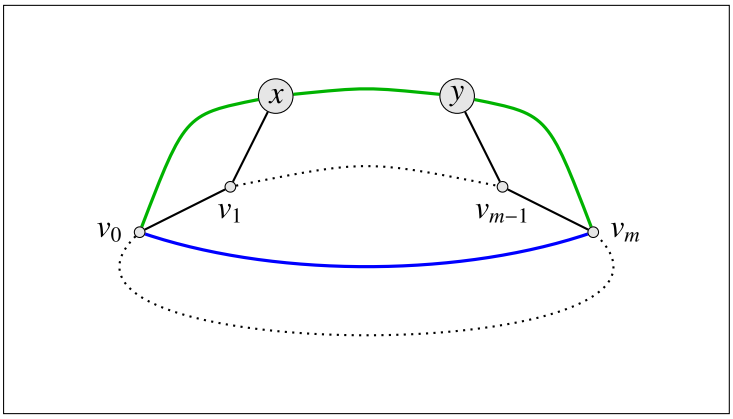

Figure 16: **Replacing $v_0 v_m$ (blue) by $v_0 x y v_m$ (green) yields a hamiltonian cycle.**

Furthermore there can be cycles of length m be appended to a vertex of C_n.

Proposition 6.5. *Let* $G = (V, E)$ *with* $V = \{v_0, \ldots, v_{n-1}, w_1, \ldots, w_{m-1}\}$. *Furthermore let* $E = \{v_0w_1, w_1w_2, \ldots, w_{m-2}w_{m-1}, w_{m-1}v_0\} \cup \{v_iv_{i+1} \mid i = 0, \ldots, n-1\}$ *(indices are taken modulo n), then* $N_m(G)$ *is hamiltonian.*

Proof. To construct a hamiltonian cycle there are the following edges of importance:

(i) $v_{n-m+1}w_1, \ldots, v_{n-1}w_{m-1} \in E(N_m(G))$ and $w_1v_1, \ldots, w_{m-1}v_{m-1} \in E(N_m(G))$; these edges are interconnecting the vertices from both cycles.

(ii) The edges $v_{n-m+1}v_1, \ldots, v_{n-1}v_{m-1} \in E(N_m(G))$ are induced by the cycle C_n.

Therefore by replacing the edges of (ii) by the paths $v_{n-m+1}w_1v_1, \ldots, v_{n-1}w_{m-1}v_{m-1}$ of (i) yields a hamiltonian cycle,
namely $(V, \{v_iv_{i+m} \mid i = 0, \ldots, n-m\} \cup \{v_{n-m+i}w_i, w_iv_i \mid i = 1, \ldots, m-1\}$.

□

An example for constructing this hamiltonian cycle is given in Figure 17 with $m = 6$.

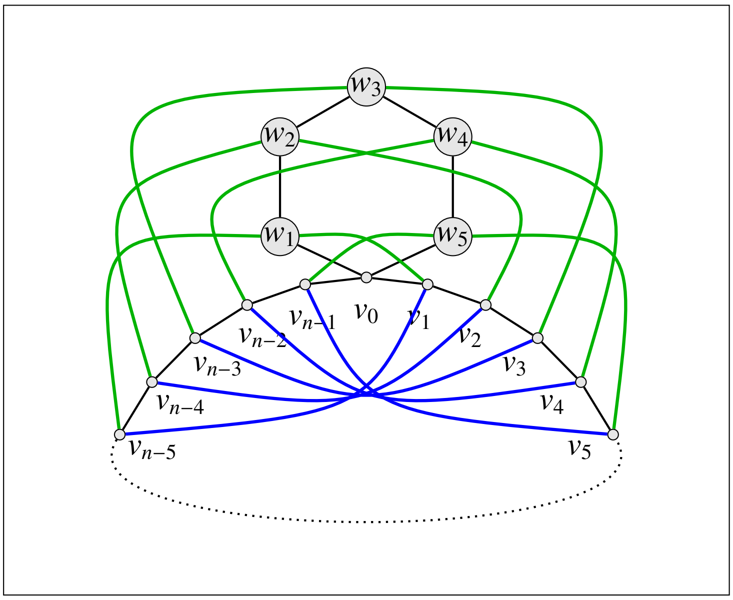

Figure 17: **Replacing** $v_{n-m+i}v_i$ **(blue) by** $v_{n-m+i}w_iv_i$ **(green),** $i = 1, \ldots, m-1$ **with** $m = 6$

Another possibilty is described in Figure 18, which is similar to the previous one using two paths of length m instead of a cycle C_m.

Proposition 6.6. *Let* $G = (V, E)$ *with* $V = \{v_0, \dots, v_{n-1}, x_1, \dots, x_{m-1}, y_1, \dots, y_{m-1}\}$. *Furthermore let* $E = \{v_0x_1, x_1x_2, \dots, x_{m-2}x_{m-1}, v_0y_1, y_1y_2, \dots, y_{m-2}y_{m-1}\} \cup \{v_iv_{i+1} \mid i = 0, \dots, n-1\}$ *(indices are taken modulo n), then* $N_m(G)$ *is hamiltonian.*

Proof. Similar to the previous proof a hamiltonian cycle is constructed by replacing the edges $v_{n-m+1}v_1, \dots, v_{n-1}v_{m-1} \in E(N_m(G))$ by the paths $v_{n-m+1}x_1y_{m-i}v_1, \dots, v_{n-1}x_{m-1}y_1v_{m-1}$, namely $(V, \{v_iv_{i+m} \mid i = 0, \dots, n-m\} \cup \{v_{n-m+i}x_i, x_iy_{m-i}, y_{m-i}v_i \mid i = 1, \dots, m-1\}$.

□

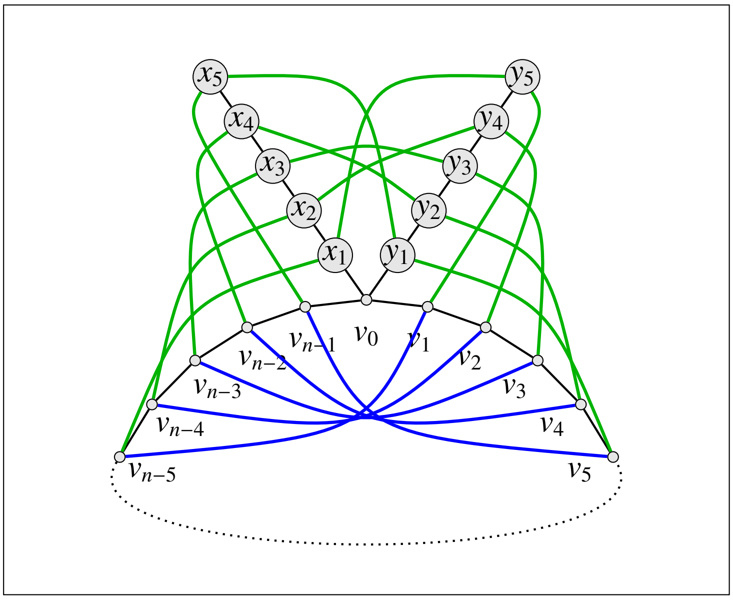

Figure 18: **Replacing** $v_{n-m+i}v_i$ **(blue) by** $v_{n-m+i}x_iy_{m-i}v_i$ **(green),** $i = 1, \dots, m-1$ **with** $m = 6$

7 Conclusions

The m-step graph is an interesting theoretical construct and closely related to the topical research on neighborhood graphs, competition graphs, m-step competition graphs and their other generalizations. Describing the m-step graphs for particular graph classes is a straightforward procedure. The more interesting question is: How are graph properties preserved by the m-step function? The minimum degree was a perfect example; as long as $m \leq \delta(G)$, the minimum degree of the m-step graph is at least $\delta(N_m(G)) \geq \delta(G) - 1$. Other interesting graph properties could be the girth and circumference. Furthermore the results for connectivity 5 and hamiltonicity 6 are elemental results, but there are many questions left. The following section describes a small selection of open problems concerning m-step graphs.

7.1 Open Problems

- For $m \geq 3$ it was not difficult to show that P_2 together with isolated vertices is an m-step graph. P_3 was shown to be not an m-step graph, no matter how many isolated vertices are added. By that one might conjecture that even paths are m-step graphs and odd paths are not. For $m = 2$ however it is obvious that no path of length at least two is a neighborhood graph.

- Is there an elegant way of describing the m-step graph of a tree? The easy thing about trees is the existence and uniqueness of xy-paths for arbitrary $x, y \in V(G)$. However, as we have seen in Section 5 the m-step graphs of trees do not necessarily decompose for odd m, which makes an elegant description difficult.

- In Section 2.2 we have seen the history of NP-completeness for determining the competition number. However, because the proof of Opsut [15] makes heavily use of directed arcs, it is difficult and perhaps impossible to translate it to the undirected case, i.e. the embedding number.

- The isomorphism problems in Section 4.2 ask for classes of graphs fulfilling the equations $N_m(G) = K_n$, $N_m(G) = G$, $N_m(G) = \overline{G}$. However, no elegant descriptions of the graphs fulfilling these equations are known yet. Another interesting problem could arises when comparing $N_m^2 = N_m(N_m(G))$ with $N_{2m}(G)$, or in general $N_m^k(G) = N_{m \cdot k}(G)$?

References

[1] B. D. Acharya and M. N. Vartak. Open neighborhood graphs. Indian Institute of Technology, Bombay, Department of Mathematics, Research Report No. 7. (1973).

[2] James W. Boland, Robert C. Brigham, and Ronald D. Dutton. The difference between a neighborhood graph and a wheel. Congressus Numerantium 58 (1987), 151–156.

[3] James W. Boland, Robert C. Brigham, and Ronald D. Dutton. Embedding arbitrary graphs in neighborhood graphs. Journal of Combinatorics, Information and System Sciences 12 (1987), 101–112.

[4] Robert C. Brigham and Ronald D. Dutton. On neighborhood graphs. Journal of Combinatorics, Information and System Sciences 12 (1987), 75–85.

[5] Han Hyuk Cho, Suh-Ryung Kim, and Yunsun Nam. The m-step competition graph of a digraph. Discrete Applied Mathematics 105 (2000), 115–127.

[6] J. E. Cohen. Interval graphs and food webs: A finding and a problem. Document 17696-PR (1968).

[7] Stephen A. Cook. The complexity of theorem-proving procedures. Proceedings of the third annual ACM symposium on Theory of computing (1971), 151–158.

[8] Reinhard Diestel. Graph theory (graduate texts in mathematics). Springer (2005).

[9] Kathryn F. Fraughnaugh, Richard Lundgren, Sarah K. Merz, John S. Maybee, and Norman J. Pullmann. Competition graphs of strongly connected and hamiltonian digraphs. Siam J. Alg. Disc. Math. 8 (1995), 179–185.

[10] Geir T. Helleloid. Connected triangle-free m-step competition graphs. Discrete Applied Mathematics 145 (2005), 376–383.

[11] Wei Ho. The m-step, same-step, and any-step competition graphs. Discrete Applied Mathematics 152 (2005), 159–175.

[12] Richard. M. Karp. Reducibility among combinatorial problems. In R.E. Miller, J.W. Thatcher (ed.), Complexity of Computer Computations (1972), 85–103.

[13] J. Richard Lundgren, Sarah K. Merz, John S. Maybee, and Craig W. Rasmussen. A characterization of graphs with interval two-step graphs. Linear Algebra and its Applications 217 (1995), 203–223.

[14] J. Richard Lundgren, Sarah K. Merz, and Craig W. Rasmussen. Chromatic numbers of competition graphs. Linear Algebra and its Applications 217 (1995), 225–239.

[15] Robert J. Opsut. On the computation of the competition number of a graph. Siam J. Alg. Disc. Math. 3 (1982), 420–428.

[16] J. B. Orlin. Contentment in graph theory: Covering graphs with cliques. Indagationes Math (1977), 406–424.

[17] Michael R. Pfützenreuter. Competition graphs of undirected graphs, 2006. Bachelor thesis at the University of Lübeck, Institute of Mathematics.

[18] Ingo Schiermeyer, Martin Sonntag, and Hanns-Martin Teichert. Structural properties and hamiltonicity of neighborhood graphs. Preprint (2009).

[19] Martin Sonntag and Hanns-Martin Teicher. Competition hypergraphs. Discrete Applied Mathematics 143 (2004), 324–329.

[20] Martin Sonntag and Hanns-Martin Teichert. Competition hypergraphs of digraphs with certain properties, part 1, strong connectedness. Disussiones Mathematicae, Graph Theory 28 (2008), 5–21.

[21] Martin Sonntag and Hanns-Martin Teichert. Competition hypergraphs of digraphs with certain properties, part 2, hamiltonicity. Disussiones Mathematicae, Graph Theory 28 (2008), 23–34.